GLOBAL DYNAMICS ABOVE THE GROUND STATE ENERGY FOR THE CUBIC NLS EQUATION IN 3D

K. NAKANISHI AND W. SCHLAG

ABSTRACT. We extend the result in [47] on the nonlinear Klein-Gordon equation to the nonlinear Schrödinger equation with the focusing cubic nonlinearity in three dimensions, for radial data of energy at most slightly above that of the ground state. We prove that the initial data set splits into nine nonempty, pairwise disjoint regions which are characterized by the distinct behaviors of the solution for large time: blow-up, scattering to 0, or scattering to the family of ground states generated by the phase and scaling freedom. Solutions of this latter type form a smooth center-stable manifold, which contains the ground states and separates the phase space locally into two connected regions exhibiting blow-up and scattering to 0, respectively. The special solutions found by Duyckaerts, Roudenko [19], following the seminal work on threshold solutions by Duyckaerts, Merle [18], appear here as the unique one-dimensional unstable/stable manifolds emanating from the ground states. In analogy with [47], the proof combines the hyperbolic dynamics near the ground states with the variational structure away from them. The main technical ingredient in the proof is a "one-pass" theorem which precludes "almost homoclinic orbits", i.e., those solutions starting in, then moving away from, and finally returning to, a small neighborhood of the ground states. The main new difficulty compared with the Klein-Gordon case is the lack of finite propagation speed. We need the radial Sobolev inequality for the error estimate in the virial argument. Another major difference between [47] and this paper is the need to control two modulation parameters.

CONTENTS

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1. Introduction

The local well-posedness of the cubic NLS equation

$$i\partial_t u - \Delta u = |u|^2 u \tag{1.1}$$

in the energy space H^1 is classical, see Strauss [55], Sulem, Sulem [56], Cazenave [11], and Tao [57]. One has mass and energy conservation

$$\begin{aligned} M(u) &= \frac{1}{2}\|u\|_2^2 = \text{const.} \\ E(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{4}\|u\|_4^4 = \text{const.}, \end{aligned} \tag{1.2}$$

where $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^3)$ norm. Data with small H^1 norm have globally defined solutions which scatter to a free wave. In the defocusing case it is known that all energy solutions scatter to zero, see Ginibre, Velo [23], [24]. In contrast, (1.1) is known to exhibit energy data for which the solutions blow up in finite time. In fact, Glassey [25] proved that all data of negative energy are of this type provided they also have finite variance. The latter assumption was later removed in the radial case by Ogawa, Tsutsumi [48].

Eq. (1.1) possesses a family of special oscillatory solutions of the form $u(t,x) = e^{-it\alpha^2+i\theta}Q(x,\alpha)$ where $\alpha > 0$ and

$$-\Delta Q(\cdot,\alpha) + \alpha^2 Q(\cdot,\alpha) = |Q|^2 Q(\cdot,\alpha)$$

There is a unique positive, radial solution to this equation called the ground state, see Strauss [54], Berestycki, Lions [7], Coffman [13], Kwong [40]. It is characterized as the solution of minimal action. Letting modulation and Galilean symmetries act on these special solutions $u(t,x)$ generates an eight-dimensional manifold of solitons. In the radial case, the manifold is only two-dimensional.

The question of orbital stability of these solitons in the energy space was settled by Weinstein [60], [61], Berestycki, Cazenave [6], and Cazenave, Lions [12]. A general theory which covers this case was developed by Grillakis, Shatah, Strauss [28], [29]. The cut-off in the power $|u|^{p-1}u$ in the n-dimensional case turns out to be the L^2 critical one $p_0 = \frac{4}{n}+1$, with $p \geq p_0$ being unstable and $p < p_0$ stable. In particular, the cubic NLS (1.1) is unstable. Recently, Holmer, Roudenko [31] showed that for all radial solutions u with mass $\|u\|_2 = \|Q\|_2$ and energy $E(u) < E(Q)$ there is the following dichotomy: if $\|\nabla u\|_2 < \|\nabla Q\|_2$ one has global existence and scattering (as $|t| \to \infty$), whereas for $\|\nabla u\|_2 > \|\nabla Q\|_2$ there is finite time blowup in both time directions. The radial assumption was then removed in Duyckaerts, Holmer, Roudenko [17]. Note that the mass condition is easily removed by scaling, with $M(u)E(u)$ being the natural scaling-invariant version of the energy, and with $M(u)\|\nabla u\|_2^2$ replacing $\|\nabla u\|_2^2$. It follows from the variational properties of Q that these regions are invariant under the NLS flow. The methods in both papers follow the ideology of Kenig-Merle [35], [36] which in turn use the concentrated

compactness decompositions of Bahouri, Gerard [1], Merle, Vega [46], as well as Keraani [37].

In a different direction, in recent years several authors have studied *conditional asymptotic stability* for the case of (1.1) as well as other equations, see [51], [38], and Beceanu [4]. This refers to the fact that solitons remain asymptotically stable even in the unstable case provided the perturbations are chosen to lie on a manifold of finite codimension near the soliton manifold. The number of "missing" dimensions here equals the number of exponentially unstable modes of the linearized equation. In the case of NLS this number equals 1. These investigations are related to the classical notion of stable, unstable, and center-stable manifolds in dynamical systems, see Bates, Jones [2] and Gesztesy, Jones, Latushkin, Stanislavova [22] for a development of these ideas applicable to NLKG and NLS.

In this paper we find that the center-stable manifolds act as boundary between a region of finite time blow-up and one of scattering to zero. In what follows

$$\mathcal{S}_\alpha := \{e^{i\theta}Q(\cdot,\alpha) \mid \theta\in\mathbb{R}\}, \quad \mathcal{S} := \bigcup_{\alpha>0}\mathcal{S}_\alpha,$$

and we set $Q = Q(\cdot,1)$ for convenience. Then $Q(x,\alpha) = \alpha Q(\alpha x)$ and $M(Q(\cdot,\alpha)) = \alpha^{-1}M(Q)$. First, we present the following result which does not rely on the notion of a center-stable manifold. Let $\mathcal{H} = H^1_{\mathrm{rad}}(\mathbb{R}^3)$ and

$$\mathcal{H}^\varepsilon := \{u\in\mathcal{H} \mid M(u)E(u) < M(Q)(E(Q)+\varepsilon^2)\} \tag{1.3}$$

as well as

$$\mathcal{H}^\varepsilon_\alpha := \mathcal{H}^\varepsilon \cap \{u\in\mathcal{H} \mid M(u) = M(Q(\cdot,\alpha))\} \tag{1.4}$$

for any $\alpha > 0$.

Theorem 1.1. *There exists $\varepsilon > 0$ small such that all solutions of* (1.1) *with data in $\mathcal{H}^\varepsilon_1$ exhibit one of the following nine different scenarios, with each case being attained by infinitely many data in $\mathcal{H}^\varepsilon_1$:*

(1) *Scattering to 0 for both $t\to\pm\infty$,*
(2) *Finite time blowup on both sides $\pm t > 0$,*
(3) *Scattering to 0 as $t\to\infty$ and finite time blowup in $t<0$,*
(4) *Finite time blowup in $t>0$ and scattering to 0 as $t\to-\infty$,*
(5) *Trapped by $\mathcal{S}_1$ for $t\to\infty$ and scattering to 0 as $t\to-\infty$,*
(6) *Scattering to 0 as $t\to\infty$ and trapped by $\mathcal{S}_1$ as $t\to-\infty$,*
(7) *Trapped by $\mathcal{S}_1$ for $t\to\infty$ and finite time blowup in $t<0$,*
(8) *Finite time blowup in $t>0$ and trapped by $\mathcal{S}_1$ as $t\to-\infty$,*
(9) *Trapped by $\mathcal{S}_1$ as $t\to\pm\infty$,*

where "trapped by $\mathcal{S}_1$" means that the solution stays in a $O(\varepsilon)$ neighborhood of $\mathcal{S}_1$ relative to H^1 forever after some time (or before some time). The initial data sets for (1)-(4), respectively, are open in $\mathcal{H}^\varepsilon_1$. The set of data in H^1 for which the associated solutions of (1.1) *forward scatter, i.e.,* $(1)\cup(3)\cup(6)$, *is open, pathwise connected, and unbounded; in fact, it contains curves which connect* 0 *to* ∞ *in H^1.*

The reason behind the number 9 is simply that all combinations of the three possibilities at $t=+\infty$ (blowup, scattering, trapping) and the corresponding ones at $t=-\infty$ are allowed. The theorem applies to solutions of any mass by rescaling.

More precisely, if $u \in \mathcal{H}^\varepsilon_\alpha$, then the statement remains intact with $\mathcal{S}_1$ replaced by $\mathcal{S}_\alpha$ and "trapped" by $\mathcal{S}_\alpha$ now meaning that $\mathrm{dist}(u, \mathcal{S}_\alpha) \lesssim \varepsilon$ where the distance is measured in the metric

$$\| \cdot \|_{H^1_\alpha} := \Big(\alpha^{-1} \| \cdot \|^2_{\dot H^1} + \alpha \| \cdot \|_2^2 \Big)^{\frac{1}{2}}. \tag{1.5}$$

As in [47], the main novel ingredient is the "one-pass theorem", see Theorem 4.1 below. It precludes almost homoclinic orbits which start very close to $\mathcal{S}_1$ and eventually return very close to $\mathcal{S}_1$. In combination with an analysis of the hyperbolic dynamics near $\mathcal{S}_1$ which results from the exponentially unstable nature of the ground state solution, this allows one to show that the fate of the solution is governed by a virial-type functional K after it exits a neighborhood of $\mathcal{S}_1$.

Using some finer spectral properties of the Hamiltonian obtained by linearizing the NLS equation around Q, see Proposition B.1, we can formulate the following stronger statement which describes in more detail what "trapping" means. In this case it is better not to freeze the mass. In other words, we work with the full set $\mathcal{H}^\varepsilon$. We require the following terminology:

Definition 1.1. Let $u(0) \in \mathcal{H}^\varepsilon$ define a solution $u(t)$ of (1.1) for all $t \geq 0$. We say that u *forward scatters to* $\mathcal{S}$ iff there exist continuous curves $\theta : [0, \infty) \to \mathbb{R}$ and $\alpha : [0, \infty) \to (0, \infty)$, as well as $u_\infty \in \mathcal{H}$ such that for all $t \geq 0$

$$u(t) = e^{i\theta(t)} Q(\cdot, \alpha(t)) + e^{-it\Delta} u_\infty + \Omega(t) \tag{1.6}$$

where $\|\Omega(t)\|_{H^1} \to 0$ as $t \to \infty$, $\alpha(t) \to \alpha_\infty > 0$ as $t \to \infty$.

Note that one then necessarily has

$$\begin{aligned} M(u) &= M(Q(\cdot, \alpha_\infty)) + M(u_\infty) = \alpha_\infty^{-1} M(Q) + M(u_\infty) \\ E(u) &= E(Q(\cdot, \alpha_\infty)) + \frac{1}{2} \|\nabla u_\infty\|_2^2 = \alpha_\infty E(Q) + \frac{1}{2} \|\nabla u_\infty\|_2^2 \end{aligned} \tag{1.7}$$

whence (using that $E(Q) = M(Q) > 0$),

$$\alpha_\infty^{-1} \|\nabla u_\infty\|_2^2 + \alpha_\infty \|u_\infty\|_2^2 + \frac{\|u_\infty\|_2^2 \|\nabla u_\infty\|_2^2}{2M(Q)} \leq 2\varepsilon^2, \quad \frac{M(Q)}{M(u)} \leq \alpha_\infty \leq \frac{E(u)}{E(Q)}, \tag{1.8}$$

and in particular, we conclude that $\|u_\infty\|_{H^1_{\alpha_\infty}} \leq \varepsilon$, that α_∞ is bounded from both above and below, and that $M(u)E(u) \geq M(Q)E(Q)$.

The heuristic meaning of (1.6) is simply that u asymptotically decomposes into a soliton $e^{i\theta_\infty(t)} Q(\cdot, \alpha_\infty)$ plus an H^1-solution to the free Schrödinger equation (however, the phase θ_∞ is not precisely the one associated with $Q(\cdot, \alpha_\infty)$ which would mean $-t\alpha_\infty^2 + \gamma_\infty$). In fact, in those cases where we can establish (1.6) we will be able to obtain finer statements on θ and α, cf. Section 7.

Theorem 1.2. *There exists $\varepsilon > 0$ small such that all solutions of* (1.1) *with data in $\mathcal{H}^\varepsilon$ exhibit one of the nine different scenarios described in Theorem 1.1, provided we replace "trapped by $\mathcal{S}_1$" with "scattering to $\mathcal{S}$". Moreover, each case is attained by infinitely many data in $\mathcal{H}^\varepsilon$. The sets* (5) ∪ (7) ∪ (9) *and* (6) ∪ (8) ∪ (9) *are smooth codimension-one manifolds in the phase space $\mathcal{H}$. Similarly, (9) is a smooth manifold of codimension two, and it contains $\mathcal{S}$.*

Using common terminology from dynamical systems, see for example Hirsch, Pugh, Shub [30], Vanderbauwhede [59], and Bates, Jones [2], we can say that (5) $\cup$ (7) $\cup$ (9) and (6) $\cup$ (8) $\cup$ (9) are the center-stable manifold $\mathcal{M}_{cs}$, resp. the center-unstable manifold $\mathcal{M}_{cu}$, associated with Q — *modulo the symmetries* given by α and θ. Since center manifolds are in general not unique it might be more precise to say "a center-stable manifold" here. However, our manifolds are naturally unique for the global characterization in Theorem 1.1. Similarly, (9) is the center manifold of Q, again modulo the symmetries given by α and θ.

Every point $p \in \mathcal{S}$ has a neighborhood $B_\varepsilon(p)$ of size $\lesssim \varepsilon$ relative to the metric (1.5) with $\alpha = M(Q)/M(p)$, such that $B_\varepsilon(p)$ is divided by $\mathcal{M}_{cs}$ into two connected components; all data in one component lead to finite time blow-up for positive times, whereas all data in the other lead to global solutions for positive times which scatter to zero as $t \to +\infty$. All solutions starting on $\mathcal{M}_{cs}$ itself scatter to $\mathcal{S}$ in the sense of (1.6) as $t \to +\infty$.

The study of stable/unstable/center-stable manifolds near equillibria of ODEs (also in infinite dimensions) has a long history in dynamics. In fact, their existence for the cubic NLS (1.1) was shown in [22] and [2]. However, in contrast to Theorem 1.2 no results are obtained there concerning the long-time behavior of the solutions on the center manifold. The unique (up to the modulation and dilation symmetries) one-dimensional stable/unstable manifolds emanating from Q are characterized by the requirement that $u(t) \to e^{-it}Q$ in H^1 exponentially fast as $t \to \infty$ or $t \to -\infty$. Clearly, the corresponding solutions must have energy equal to that of Q. The same definition applies to $\mathcal{S}$ with Q being replaced by $e^{-it\alpha^2+i\theta_0}Q(\cdot,\alpha)$. In our work these one-dimensional manifolds (up to the symmetries) appear naturally in the form of those solutions found by Duyckaerts, Roudenko [19]. It is important to note that we can therefore completely describe the global (i.e., both as $t \to \infty$ as well as $t \to -\infty$) behavior of the stable/unstable manifolds in this setting.

Theorem 1.3. *Consider the limiting case $\varepsilon \to 0$ in Theorem 1.1, i.e., all the radial solutions satisfying $E(u) \le E(Q)$ and $M(u) = M(Q)$. Then the sets (3) and (4) vanish, while the sets (5)-(9) are characterized, with some special solutions $W_\pm$ of* (1.1)*, as follows:*

$$\begin{aligned}
&(5) = \{e^{i\theta}W_-(t-t_0) \mid t_0,\theta \in \mathbb{R}\}, \quad (6) = \{e^{i\theta}\overline{W_-}(-t-t_0) \mid t_0,\theta \in \mathbb{R}\},\\
&(7) = \{e^{i\theta}W_+(t-t_0) \mid t_0,\theta \in \mathbb{R}\}, \quad (8) = \{e^{i\theta}\overline{W_+}(-t-t_0) \mid t_0,\theta \in \mathbb{R}\},\\
&(9) = \{e^{-i(t+\theta)}Q \mid \theta \in \mathbb{R}\}.
\end{aligned} \tag{1.9}$$

The sets (5)$\cup$(7)$\cup$(9) *form the stable manifold, whereas* (6)$\cup$(8)$\cup$(9) *are the unstable manifold of Q, up to the modulation symmetry. In other words, solutions in* (5), (7) *and* (6), (8) *approach a soliton trajectory in $\mathcal{S}_1$ exponentially fast as $t \to \infty$ or $t \to -\infty$, respectively. An analogous statement holds without the mass constraint, but then these sets take the form $\{e^{i\theta}\alpha W_\pm(\alpha^2(t-t_0),\alpha x)\}$, $\{e^{i\theta}\alpha\overline{W_\pm}(-\alpha^2(t+t_0),\alpha x)\}$, resp. $\{e^{-i(t\alpha^2+\theta)}Q(\cdot,\alpha)\}$, where θ,α vary.*

This paper is organized as follows. In Section 2 we review some variational properties of the ground state and discuss the linearized operators. In Section 3 we present the modulation method which we use in the proof of Theorem 1.1. Since Theorem 1.1 is closer to orbital stability than asymptotic stability, the modulation

approach of Section 3 is less precise but easier to work with than the one usually employed in asymptotic stability theory. Section 4 presents the one-pass theorem, and is of central importance to the entire paper. The proof of that theorem is more involved than in the Klein-Gordon case [47], due to the lack of finite speed of propagation. We will modify Ogawa-Tsutsumi's *saturated virial* identity [48] in the radial energy space, fitting it in the variational argument away from the ground state. Section 5 shows by a Kenig-Merle type argument [35], that those solutions which are guaranteed by the one-pass theorem to exist for all positive times actually scatter to zero. The proof of Theorem 1.1 is then given in Section 6. Up until that point, our arguments do not require any fine spectral properties of the linearized NLS Hamiltonian. This changes in Section 7 where we construct the center-stable manifold in the radial energy class near Q following the method in [51] and [5] (we remark that Beceanu [5] has constructed the manifold in $\dot{H}^{\frac{1}{2}}$ without any radial assumption). Some of the aforementioned spectral properties are – at least for the moment – only known via numerically assisted arguments, see [16] and [41] as well as Proposition B.1. More precisely, for the structure of the real spectrum we rely on the recent work of Marzuola and Simpson [41] which is partially numerical (in the spirit of Fibich, Merle, Raphael [21]). The construction of the manifold relies on a novel dispersive estimate due to Beceanu [4] which allows for *small but not decaying* and purely time-dependent lower-order perturbations to a Schrödinger operator. We rederive what is needed from [4] in our setting in Section B. Section 8 presents the proofs of Theorems 1.2 and 1.3, and they require the center-stable manifold of Section 7. Section A recalls some basic results related to the scattering theory of (1.1) such as the Bahouri-Gerard decomposition in this setting, and the perturbation lemma needed for the Kenig-Merle method, and Section C gives a proof for some radial Sobolev-type inequalities.

The research in this paper as well as that of [47] is part of the wider area encompassing dispersive equations and their global existence theory on the one hand, and the theory of unstable equilibria such as the ground state soliton on the other hand. Especially for the L^2 critical NLS equation substantial progress has been made on the very delicate blowup phenomena exhibited at and near the ground state. The L^2 critical equation is special due to its invariance under the *pseudo-conformal transformation*, see for example [11]. Applying this class of transformations to the ground state Q gives rise to a solution blowing up in finite time, and it is unique with this property at exactly the mass of Q, see Merle [42]. Bourgain, Wang [8] studied the *conditional stability* of the pseudo-conformal blowup on a submanifold of large codimension, and Krieger and the second author [39] established the existence of a codimension 1 submanifold (albeit with no regularity and in a strong topology) for which these solutions are preserved. The conjecture that the pseudo-conformal should be stable under a codimension 1 condition is due to Galina Perleman [49].

A sweeping analysis of the *stable* blowup regime near the ground state for the L^2-critical case was carried out by Merle, Raphaël [43] in a series of works, preceded by [49] which established the existence of the so-called log log blowup regime. Very recently Merle, Raphaël, and Szeftel [45] proved that the Bourgain-Wang solutions are on the threshold between the log log blowup and the scattering regimes. In [44] Merle, Raphaël and Szeftel were able to transfer some of the techniques from

the critical case to the slightly L^2-supercritical one and established stable blowup dynamics near the ground state in that case.

The L^2-critical instability of the ground state is *algebraic* in nature rather than exponential, and thus very far from the considerations in this paper. We emphasize that the hyperbolic dynamics is exploited strongly in our arguments. In addition, we rely heavily on the *radial* assumption, for example in the virial argument.

2. The ground state and the linearized operator

In this section we recall some variational and spectral properties around the ground states. The scaled family of ground states $Q(\alpha) = Q_\alpha := \alpha Q(\alpha x)$ solves

$$\begin{aligned} &-\Delta Q_\alpha + \alpha^2 Q_\alpha = Q_\alpha^3, \\ &\|\nabla Q_\alpha\|_2^2 = \alpha \|\nabla Q\|_2^2, \quad \|Q_\alpha\|_4^4 = \alpha \|Q\|_4^4, \quad \|Q_\alpha\|_2^2 = \alpha^{-1}\|Q\|_2^2. \end{aligned} \tag{2.1}$$

Differentiating in α yields

$$(-\Delta + \alpha^2 - 3Q_\alpha^2)Q'_\alpha = -2\alpha Q_\alpha. \tag{2.2}$$

The relevant functionals in this paper are defined as

$$\begin{aligned} &E(u) = \|\nabla u\|_2^2/2 - \|u\|_4^4/4, \quad M(u) = \|u\|_2^2/2, \\ &J(u) = \|\nabla u\|_2^2/2 + \|u\|_2^2/2 - \|u\|_4^4/4, \\ &K(u) = \|\nabla u\|_2^2 - \frac{3}{4}\|u\|_4^4, \end{aligned} \tag{2.3}$$

the first three being the conserved energy, mass, and action, respectively. The functional K results from pairing $J'(u)$ with $(x\nabla + \nabla x)u/2$, the generator of dilations. By construction, Q is a critical point of J, i.e., $J'(Q) = 0$ whence also $K(Q) = 0$. Moreover, the region

$$M(u)E(u) < M(Q)E(Q) \tag{2.4}$$

is divided into two connected components by the conditions $\{K \geq 0\}$ and $\{K < 0\}$. The quantity ME in (2.4) is scaling invariant and was used by Holmer, Roudenko [31] in their scattering analysis. The aforementioned division into two connected components is intimately linked to the following minimization property. Define positive functional G and I by

$$\begin{aligned} &G(\varphi) := J(\varphi) - \frac{K(\varphi)}{3} = \frac{1}{6}\|\nabla \varphi\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^2}^2, \\ &I(\varphi) := J(\varphi) - \frac{K(\varphi)}{2} = \frac{1}{2}\|\varphi\|_{L^2}^2 + \frac{1}{8}\|\varphi\|_{L^4}^4. \end{aligned} \tag{2.5}$$

Lemma 2.1. *We have*

$$\begin{aligned} J(Q) &= \inf\{J(\varphi) \mid 0 \neq \varphi \in H^1,\ K(\varphi) = 0\} \\ &= \inf\{G(\varphi) \mid 0 \neq \varphi \in H^1,\ K(\varphi) \leq 0\} \\ &= \inf\{I(\varphi) \mid 0 \neq \varphi \in H^1,\ K(\varphi) \leq 0\}, \end{aligned} \tag{2.6}$$

and these infima are achieved only by $e^{i\theta}Q(x - c)$, with $\theta \in \mathbb{R}$ and $c \in \mathbb{R}^3$.

For the proof, see for example [47, Lemma 2.1] and [33, Lemma 2.3]. In particular, $J(\varphi) < J(Q)$ implies either $\varphi = 0$, $K(\varphi) > 0$ or $K(\varphi) < 0$.

Next, consider a decomposition of the solution in the form

$$u = e^{i\theta}(Q + w). \tag{2.7}$$

Inserting this into NLS yields

$$\begin{aligned} i\dot{w} &= e^{-i\theta}(\Delta u + |u|^2 u + \dot{\theta} u) \\ &= (\Delta + \dot{\theta})(Q + w) + |Q|^2 Q + 2|Q|^2 w + Q^2 \overline{w} + 2Q|w|^2 + w^2 \overline{Q} + |w|^2 w \\ &= (1 + \dot{\theta})(Q + w) - \mathcal{L}w + N(w), \end{aligned} \tag{2.8}$$

The $\mathbb{R}$-linear operator $\mathcal{L}$ defined by[1]

$$\mathcal{L}w := -\Delta w + w - 2Q^2 w - Q^2 \overline{w}, \tag{2.9}$$

is self-adjoint on $L^2(\mathbb{R}^3; \mathbb{C})$ with the inner product

$$\langle f|g\rangle := \operatorname{Re} \int_{\mathbb{R}^3} f(x)\overline{g(x)}dx, \tag{2.10}$$

and $N(w)$ is the nonlinear part defined by

$$N(w) = 2Q|w|^2 + Qw^2 + |w|^2 w. \tag{2.11}$$

Note that $i\mathcal{L}$ is symmetric with respect to the symplectic form

$$\Omega(f, g) := \operatorname{Im} \int_{\mathbb{R}^3} \overline{f(x)} g(x) dx = \langle if|g\rangle$$

i.e., $\Omega(i\mathcal{L}f, g) = \Omega(i\mathcal{L}g, f)$. The generalized eigenfunctions of $i\mathcal{L}$ are as follows:

$$i\mathcal{L}iQ = 0, \quad i\mathcal{L}Q' = -2iQ, \quad i\mathcal{L}\mathfrak{G}_\pm = \pm\mu\mathfrak{G}_\pm, \tag{2.12}$$

where $\mu > 0$,

$$Q' = \partial_\alpha Q_\alpha|_{\alpha=1} = (1 + r\partial_r)Q, \quad \mathfrak{G}_\pm = \varphi \mp i\psi, \tag{2.13}$$

and with φ, ψ real-valued. In terms of the real and imaginary values, these equations are

$$L_- Q = 0, \quad L_+ Q' = -2Q, \quad L_- \psi = \mu\varphi, \quad L_+ \varphi = -\mu\psi \tag{2.14}$$

with

$$L_- = -\Delta + 1 - Q^2, \qquad L_+ = -\Delta + 1 - 3Q^2 \tag{2.15}$$

The existence of φ, ψ is standard and follows from the minimization

$$\min\{\langle \sqrt{L_-} L_+ \sqrt{L_-} f|f\rangle \mid \|f\|_2^2 \le 1\} < 0 \tag{2.16}$$

Recall that $L_- \ge 0$ and $\ker(L_-) = \{Q\}$. In other words, $\langle L_- f|f\rangle \gtrsim \|f\|_{H^1}^2$ if $f \perp Q$. After appropriate normalization of (φ, ψ), we have

$$\begin{aligned} &\langle iiQ|Q'\rangle = -\langle Q|Q'\rangle = M(Q), \quad \langle i\mathfrak{G}_+|\mathfrak{G}_-\rangle = 2\langle\varphi|\psi\rangle = 2\langle L_-\psi|\psi\rangle/\mu = 1, \\ &0 = \langle Q|\mathfrak{G}_\pm\rangle = \langle iQ'|\mathfrak{G}_\pm\rangle = \langle\varphi|Q\rangle = \langle\psi|Q'\rangle. \end{aligned} \tag{2.17}$$

[1]We need not extend $\mathcal{L}$ as a $\mathbb{C}$-linear operator until Section 7, where we introduce a different notation. Hence the linear algebra for $\mathcal{L}$ is always carried out in the sense of an $\mathbb{R}$-vector space.

Moreover $\langle\psi|Q\rangle \neq 0$ and so we can choose $\langle\psi|Q\rangle > 0$. To see this, suppose $\psi \perp Q$, then $\varphi \perp L_+Q = -2Q^3$, and so by Lemma 2.3 of [47], $0 \leq \langle L_+\varphi|\varphi\rangle = -\mu\langle\psi|\varphi\rangle < 0$, which is a contradiction.

The symplectic decomposition of $L^2(\mathbb{R}^3;\mathbb{C})$ corresponding to these discrete modes is uniquely given by

$$\begin{aligned} &f = aiQ + bQ' + c_+\mathfrak{G}_+ + c_-\mathfrak{G}_- + \eta, \\ &a = \langle if|Q'\rangle/M(Q), \quad b = -\langle f|Q\rangle/M(Q), \quad c_\pm = \pm\langle if|\mathfrak{G}_\mp\rangle, \end{aligned} \tag{2.18}$$

One has $0 = \langle\eta|Q\rangle = \langle i\eta|Q'\rangle = \langle i\eta|\mathfrak{G}_\pm\rangle$ and the symplectic projections onto $\{iQ, Q'\}^{i\perp}$ and onto $\{\mathfrak{G}_\pm\}^{i\perp}$ commute.

We apply the symplectic decomposition to w. Then, writing $\gamma = aiQ + bQ' + \eta$ one has

$$u = e^{i\theta}(Q + w) = e^{i\theta}(Q + \lambda_+\mathfrak{G}_+ + \lambda_-\mathfrak{G}_- + \gamma) \tag{2.19}$$

The justification for including the "root"-part (i.e., the zero modes) in γ follows from a suitable choice of the symmetry parameters α, θ, see Section 3. The action is expanded as

$$J(u) - J(Q) = \frac{1}{2}\langle\mathcal{L}w|w\rangle - C(w) = -\mu\lambda_+\lambda_- + \frac{1}{2}\langle\mathcal{L}\gamma|\gamma\rangle - C(w), \tag{2.20}$$

where the superquadratic part $C(w)$ is defined by

$$C(w) = \langle|w|^2w|Q\rangle + \|w\|_4^4/4. \tag{2.21}$$

The following lemma will guarantee the positivity of the γ component in (2.20).

Lemma 2.2. *Let $f, g \neq 0$ be real-valued, radial and satisfy*

$$\langle f|\psi\rangle = 0 = \langle g|Q'\rangle. \tag{2.22}$$

Then $\langle L_+f|f\rangle \simeq \|f\|_{H^1}^2$ and $\langle L_-g|g\rangle \simeq \|g\|_{H^1}^2$.

Proof. Let $f \neq 0$ satisfy $f \perp \psi$ and $\langle L_+f|f\rangle \leq 0$. Then

$$\langle L_+f|\varphi\rangle = \langle f| - \mu\psi\rangle = 0, \quad \langle L_+\varphi|\varphi\rangle = -\mu\langle\psi|\varphi\rangle = -\mu/2 < 0, \tag{2.23}$$

so f and φ are not colinear, and moreover

$$\langle L_+(af + b\varphi)|af + b\varphi\rangle = a^2\langle L_+f|f\rangle + b^2\langle L_+\varphi|\varphi\rangle + 2ab\langle L_+f|\varphi\rangle \leq 0 \tag{2.24}$$

for any $a, b \in \mathbb{R}$, which contradicts the fact that L_+ has only one nonpositive eigenvalue (cf. for example, Lemma 2.3 in [47]).

Next, apply the orthogonal projection of Q to g:

$$g = cQ + g', \quad g' \perp Q, \tag{2.25}$$

then $\langle g|Q'\rangle = 0$ implies that $c = -\langle g'|Q'\rangle/\langle Q|Q'\rangle$. Hence

$$\|g\|_{H^1}^2 \simeq c^2 + \langle L_-g'|g'\rangle \lesssim \|g'\|_{H^1}^2 \simeq \langle L_-g|g\rangle \tag{2.26}$$

as desired. □

The spectrum of L_- in $L^2_{\rm rad}$ consists of 0 as a ground state (simple eigenvalue), $[1,\infty)$ as essential spectrum (which is absolutely continuous); L_+ (again over the radial subspace) has a ground state with eigenvalue $-k^2<0$, no other eigenvalues in $(-k^2,\varepsilon)$ where $\varepsilon>0$, and the same essential spectrum as L_-. These properties are well-known and easy to obtain via variational arguments, see for example [47, Lemma 2.1]. More delicate is the question of eigenvalues in the gap $(0,1]$ and what the behavior is at the threshold 1. This question turns out to be irrelevant for the proof of Theorem 1.1, but is relevant once the center-stable manifold comes into play, at least with the approach that is implemented here (Lyapunov-Perron method). For the cubic nonlinearity, as it is being considered here, [16] gives numerical evidence that $L_\pm$ have no eigenvalues in $(0,1]$ and that 1 is a regular threshold (no resonance there).

3. Parameter choice

In this section, we determine the modulation parameters of the ground state part, so that we can translate the local arguments from the Klein-Gordon case [47] (where the ground state is fixed) to the modulation analysis for NLS. In particular, we will derive the ejection lemma and the variational lower bounds in the same spirit as in [47, Lemmas 4.2 and 4.3].

We determine α,θ explicitly by the equations

$$M(u)=M(Q_\alpha),\quad (u|e^{i\theta}Q_\alpha')<0, \tag{3.1}$$

where $(f|g)=\int f(x)\overline{g(x)}dx$. Indeed, both formulae can be explicitly solved by

$$\alpha=M(Q)/M(u),\quad \theta=\operatorname{Im}\log(u|-Q_\alpha'). \tag{3.2}$$

Since $(Q_\alpha|Q_\alpha')=-\alpha^{-2}M(Q)<0$, there is a unique solution $(\alpha,e^{i\theta})\in(0,\infty)\times S^1$ as long as u is close to some $e^{i\theta}Q_\alpha$. It is easy to see that $u=e^{i\varphi}Q_\beta$ gives $\varphi=\theta$ and $\alpha=\beta$. Even though this choice of parameters differs from the traditional one used in "modulation theory" (see Section 7 for the latter) we find that (3.1) is convenient for our purposes. Loosely speaking, up until Section 7 we will be working more in the spirit of orbital stability theory, whereas Section 7 requires the finer asymptotic stability property and thus a different handling of the modulation parameters.

The advantage of this choice of (α,θ) is that it is explicit and moreover $M(u)$ is conserved in time, and so α is fixed. A disadvantage is that it is nonlinear, in the sense that

$$\langle w|Q_\alpha\rangle=-M(w),\quad \langle iw|Q_\alpha'\rangle=0, \tag{3.3}$$

but this will be a higher order effect that can be ignored (we assume throughout this section that w is small). Without loss of generality we now fix

$$\alpha=1,\quad M(u)=M(Q) \tag{3.4}$$

and omit α. We can further decompose

$$u=e^{i\theta}(Q+w),\quad w=\lambda_+\mathfrak{G}_++\lambda_-\mathfrak{G}_-+\gamma,\quad \lambda_\pm=\pm\langle iw|\mathfrak{G}_\mp\rangle. \tag{3.5}$$

Moreover, define

$$\lambda_1:=(\lambda_++\lambda_-)/2,\quad \lambda_2:=(\lambda_+-\lambda_-)/2,\quad \vec\lambda:=(\lambda_1,\lambda_2). \tag{3.6}$$

so that the decomposition is written as

$$w = 2\lambda_1\varphi - 2i\lambda_2\psi + \gamma. \tag{3.7}$$

The remainder's orthogonality is given by

$$\langle\gamma|Q\rangle = -\frac{1}{2}\|w\|_2^2, \quad \langle i\gamma|Q'\rangle = 0, \quad \langle i\gamma|\mathfrak{G}_\pm\rangle = 0. \tag{3.8}$$

which, by Lemma 2.2, is sufficient for the property

$$\langle\mathcal{L}\gamma|\gamma\rangle \simeq \|\gamma\|_{H^1}^2. \tag{3.9}$$

The equation of θ is obtained by differentiating $0 = \langle iu|e^{i\theta}Q'\rangle = \langle iw|Q'\rangle$. Using the equation of w (2.8), as well as $\langle w + 2Q|w\rangle = 0$, one concludes that

$$(\dot\theta + 1)[M(Q) - \langle w|Q'\rangle] = \langle -\mathcal{L}w + N(w)|Q'\rangle = -\|w\|_2^2 + \langle N(w)|Q'\rangle. \tag{3.10}$$

The equation for $\lambda_\pm$ is obtained by differentiating (3.5). In fact,

$$\begin{aligned} \dot\lambda_\pm &= \pm\langle i\dot w|\mathfrak{G}_\mp\rangle = \langle(\dot\theta+1)(Q+w) - \mathcal{L}w + N(w)| \pm \mathfrak{G}_\mp\rangle \\ &= \pm\mu\lambda_\pm + N_\pm(w), \\ N_\pm(w) &:= \langle N(w) + (\dot\theta+1)w| \pm \mathfrak{G}_\mp\rangle, \end{aligned} \tag{3.11}$$

and so $\vec\lambda$ solves

$$\begin{aligned} \dot\lambda_1 &= \mu\lambda_2 + N_1(w), \quad N_1(w) = \langle N(w) + (\dot\theta+1)w|i\psi\rangle, \\ \dot\lambda_2 &= \mu\lambda_1 + N_2(w), \quad N_2(w) = \langle N(w) + (\dot\theta+1)w|\varphi\rangle. \end{aligned} \tag{3.12}$$

Recall the energy expansion

$$\begin{aligned} J(u) - J(Q) &= -\mu\lambda_+\lambda_- + \frac{1}{2}\langle\mathcal{L}\gamma|\gamma\rangle - C(w) \\ &= \mu[\lambda_2^2 - \lambda_1^2] + \frac{1}{2}\langle\mathcal{L}\gamma|\gamma\rangle - C(w). \end{aligned} \tag{3.13}$$

We therefore define the linearized energy norm to be

$$\|v\|_E^2 := \mu|\vec\lambda|^2 + \frac{1}{2}\langle\mathcal{L}\gamma|\gamma\rangle = \frac{\mu}{2}(\lambda_+^2 + \lambda_-^2) + \frac{1}{2}\langle\mathcal{L}\gamma|\gamma\rangle \simeq \|v\|_{H^1}^2, \tag{3.14}$$

where we used Lemma 2.2 for the final step. Furthermore, we define the smooth nonlinear distance function in such a way that, still under the mass constraint $M(u) = M(Q)$,

$$\begin{aligned} &d_Q^2(u) \simeq \inf_{\beta\in\mathbb{R}} \|u - e^{i\beta}Q\|_{H^1}^2 \\ &d_Q^2(u) = \|w\|_E^2 - \chi(\|w\|_E/(2\delta_E))C(w) \text{ if } d_Q(u) \ll 1, \end{aligned} \tag{3.15}$$

where $\delta_E \ll 1$ is chosen such that

$$\|v\|_E \le 4\delta_E \implies |C(v)| \le \|v\|_E^2/2. \tag{3.16}$$

The smooth cut-off $\chi(r)$ is equal to one on $|r| \le 1$ and vanishes for $|r| \ge 2$. To see the consistency of the above two properties, let $u = e^{i\beta}Q + v$ be a minimizer for

$$\operatorname{dist}_{H^1}(u, \mathcal{S}_1) = \inf_\beta \|u - e^{i\beta}Q\|_{H^1}.$$

Then $\langle w|iQ'\rangle = 0$ implies that $\langle e^{-i\theta}v|iQ'\rangle = \sin(\beta-\theta)M(Q)$, and so

$$\|v\|_{H^{-1}} \gtrsim \inf_{k\in\mathbb{Z}} |\beta-\theta+k\pi|$$

as long as v is small. The case of k odd can be eliminated here via the sign in (3.1). Indeed, by the second condition in (3.1),

$$-\cos(\beta-\theta)(Q|Q') > \mathrm{Re}(v|e^{i\theta}Q') \tag{3.17}$$

which excludes that $\beta-\theta$ lies near an odd multiple of π. Therefore,

$$\|w\|_E \simeq \|w\|_{H^1} \lesssim \|v\|_{H^1} \le \|w\|_{H^1}. \tag{3.18}$$

By the same argument, if $u = e^{i\beta}Q + v$ is an L^2 distance minimizer, then

$$|\beta-\theta| \lesssim \mathrm{dist}_{L^2}(u,\mathcal{S}_1), \tag{3.19}$$

provided that the right-hand side is small enough.

In the region $d_Q(u) \ll 1$, the distance function d_Q enjoys the following properties:

$$\begin{aligned} &\|w\|_E^2/2 \le d_Q^2(u) \le 2\|w\|_E^2, \quad d_Q^2(u) = \|w\|_E^2 + O(\|w\|_E^3), \\ &d_Q(u) \le \delta_E \implies d_Q^2(u) = J(u) - J(Q) + 2\mu\lambda_1^2. \end{aligned} \tag{3.20}$$

Hence as long as $d_Q(u) < \delta_E$ we have

$$\partial_t d_Q^2(u) = 4\mu\lambda_1\dot{\lambda}_1 = 4\mu^2\lambda_1\lambda_2 + 4\mu\lambda_1 N_1(w). \tag{3.21}$$

Lemma 3.1. *For any $u \in \mathcal{H}_1$ satisfying*

$$J(u) < J(Q) + d_Q(u)^2/2, \quad d_Q(u) \le \delta_E, \tag{3.22}$$

one has $d_Q(u) \simeq |\lambda_1| = -\mathfrak{s}\lambda_1$ for $\mathfrak{s} = \pm 1$.

Proof. (3.20) yields

$$d_Q^2(u) = J(u) - J(Q) + 2\mu\lambda_1^2 < d_Q^2(u)/2 + 2\mu\lambda_1^2. \tag{3.23}$$

and so, $\mu\lambda_1^2/2 \le \|w\|_E^2/2 \le d_Q^2(u) < 4\mu\lambda_1^2$. The second inequality sign uses (3.20), whereas the final one uses (3.23). □

It will be convenient to relate $d_Q(u)$ to the L^2 distance, taking advantage of the H^1 subcriticality of our nonlinearity.

Lemma 3.2. *Let $u \in \mathcal{H}_1$ satisfy*

$$\|u\|_{H^1} \lesssim 1, \quad J(u) - J(Q) \ll \delta_E^2, \quad J(u) - J(Q) < d_Q(u)^2/2. \tag{3.24}$$

Then we have

$$\mathrm{dist}_{L^2}(u,\mathcal{S}_1) = \inf_{\beta\in\mathbb{R}} \|u - e^{i\beta}Q\|_2 \gtrsim \min(d_Q(u), \delta_E^2). \tag{3.25}$$

Proof. Consider the decomposition (2.19) such that $\|w\|_2 \simeq \mathrm{dist}_{L^2}(u,\mathcal{S}_1)$. This is legitimate by (3.19). We may assume $\mathrm{dist}_{L^2}(u,\mathcal{S}_1) \ll \delta_E^2$. Then using Gagliardo-Nirenberg, we obtain

$$|C(w)| \lesssim \|w\|_2 \ll \delta_E^2, \quad |\lambda_\pm| \lesssim \|w\|_2 \ll \delta_E^2, \tag{3.26}$$

and so from (2.20),

$$\|\gamma\|_{H^1_x}^2 \simeq \frac{1}{2}\langle \mathcal{L}\gamma|\gamma\rangle = J(u) - J(Q) + \mu\lambda_+\lambda_- + C(w) \ll \delta_E^2, \tag{3.27}$$

hence $d_Q(u) \lesssim \|w\|_{H^1_x} \lesssim |\lambda_+| + |\lambda_-| + \|\gamma\|_{H^1_x} \ll \delta_E$. Then the previous lemma implies that $d_Q(u) \sim |\lambda_1|$, whence $d_Q(u) \lesssim \operatorname{dist}_{L^2}(u, \mathcal{S}_1)$ as desired. □

The following lemma exhibits the mechanism by which solutions are ejected along the unstable mode.

Lemma 3.3. *There exists a constant $0 < \delta_X \le \delta_E$, as well as constants $C_*, T_* \simeq 1$ with the following properties: Let $u(t)$ be a local solution of NLS in $\mathcal{H}_1$ on an interval $[0,T]$ satisfying*

$$R := d_Q(u(0)) \le \delta_X, \quad J(u) < J(Q) + R^2/2 \tag{3.28}$$

and for some $t_0 \in (0,T)$,

$$d_Q(u(t)) \ge R \quad (0 < \forall t < t_0). \tag{3.29}$$

Then u extends as long as $d_Q(u(t)) \le \delta_X$, and satisfies $\forall\, t \ge 0$

$$\begin{aligned} &d_Q(u(t)) \simeq -\mathfrak{s}\lambda_1(t) \simeq -\mathfrak{s}\lambda_+(t) \simeq e^{\mu t}R, \\ &|\lambda_-(t)| + \|\gamma(t)\|_E \lesssim R + e^{2\mu t}R^2, \\ &\mathfrak{s}K(u(t)) \gtrsim (e^{\mu t} - C_*)R, \end{aligned} \tag{3.30}$$

where $\mathfrak{s} = +1$ or $\mathfrak{s} = -1$ is constant. Moreover, $d_Q(u(t))$ is increasing for $t \ge T_* R$, and $|d_Q(u(t)) - R| \lesssim R^3$ for $0 \le t \le T_* R$.

Proof. Lemma 3.1 yields $d_Q(u) \simeq -\mathfrak{s}\lambda_1$ with $\mathfrak{s} = \pm 1$ fixed, as long as $R \le d_Q(u) \le \delta_E$. The exiting condition (3.29) implies $\partial_t d_Q(u)^2|_{t=0} \ge 0$. Since $|N_1(w)| \lesssim \|w\|_{H^1}^2 \lesssim \lambda_1^2$, we deduce from (3.21) that $-\mathfrak{s}\lambda_2(0) \gtrsim -|\lambda_1(0)|^2$ and so $\lambda_+(0) \simeq \lambda_1(0)$.

Integrating the equation for $\lambda_\pm$ yields

$$|\lambda_\pm(t) - e^{\pm\mu t}\lambda_\pm(0)| \lesssim \int_0^t e^{\mu(t-s)}|N_\pm(w(s))|ds \lesssim \int_0^t e^{\mu(t-s)}|\lambda_1(s)|^2 ds, \tag{3.31}$$

from which by continuity in time we deduce that as long as $Re^{\mu t} \ll 1$,

$$\lambda_1(t) \simeq \lambda_+(t) \simeq -\mathfrak{s}Re^{\mu t}, \quad |\lambda_\pm(t) - e^{\pm\mu t}\lambda_\pm(0)| \lesssim R^2 e^{2\mu t}. \tag{3.32}$$

Now consider the nonlinear energy projected onto the $\mathfrak{G}_\pm$ plane:

$$E_{\mathfrak{G}}(\lambda) := -\mu\lambda_+\lambda_- - C(\lambda_+\mathfrak{G}_+ + \lambda_-\mathfrak{G}_-), \tag{3.33}$$

where $C(\cdot)$ is defined in (2.21). Using the equation of $\lambda_\pm$, we obtain

$$\begin{aligned} \partial_t E_{\mathfrak{G}} &= -\mu\lambda_+\dot{\lambda}_- - \mu\lambda_-\dot{\lambda}_+ - \langle N(\lambda_+\mathfrak{G}_+ + \lambda_-\mathfrak{G}_-)|\dot{\lambda}_+\mathfrak{G}_+ + \dot{\lambda}_-\mathfrak{G}_-\rangle \\ &= \langle N(w) - N(\lambda_+\mathfrak{G}_+ + \lambda_-\mathfrak{G}_-)|\dot{\lambda}_+\mathfrak{G}_+ + \dot{\lambda}_-\mathfrak{G}_-\rangle \\ &\qquad + (\dot{\theta}+1)\langle w|\dot{\lambda}_+\mathfrak{G}_+ + \dot{\lambda}_-\mathfrak{G}_-\rangle \\ &\lesssim \|\gamma\|_{H^1}^2|\lambda| + |\lambda|^4. \end{aligned} \tag{3.34}$$

Hence $|\partial_t(J(u) - E_{\mathfrak{G}})| \lesssim \|\gamma\|_{H^1}^2|\lambda| + |\lambda|^4$, while

$$\begin{aligned} J(u) - J(Q) - E_{\mathfrak{G}} &= \langle \mathcal{L}\gamma|\gamma\rangle/2 - C(w) + C(\lambda_+\mathfrak{G}_+ + \lambda_-\mathfrak{G}_-) \\ &\simeq \|\gamma\|_{H^1}^2 + O(\|\gamma\|_{H^1}|\lambda|^2), \end{aligned} \tag{3.35}$$

and so

$$\|\gamma\|^2_{L^\infty_t H^1(0,T)} \lesssim \|\gamma(0)\|^2_{H^1} + \|\gamma\|_{L^\infty_t H^1(0,T)}\|\lambda\|^2_{L^\infty(0,T)} + \|\lambda\|^4_{L^4(0,T)}, \tag{3.36}$$

which is sufficient. Indeed, the desired estimate on γ follows from (3.32) inserted into (3.36). The equation of λ_2 implies $-\mathfrak{s}\lambda_2(t) \gtrsim R(e^{\mu t} - 1) - O(R^2)$, hence there is $T_* \simeq 1$ such that $-\mathfrak{s}\lambda_2 \gtrsim R$ and $\partial_t d_Q(u) > 0$ for $t \geq T_* R$. For $0 \leq t \leq T_* R$, we have $|\partial_t d_Q(u)| \lesssim R^2$ and so $|d_Q(u) - R| \lesssim R^3$.

Finally, we expand K around Q:

$$\begin{aligned} K(u) = K(Q + w) &= \langle -2Q + L_+Q/2|w_1\rangle + O(\|w\|^2_{H^1}) \\ &= -\lambda_1\mu\langle Q|\psi\rangle - \langle 2Q + Q^3|\gamma\rangle + O(\|w\|^2_{H^1}). \end{aligned} \tag{3.37}$$

Since $\langle Q|\psi\rangle > 0$, we obtain the desired bound on K from the behavior of λ_1. $\square$

The following lemma gives lower bounds on $|K|$, which should be used once the solution is away from $\mathcal{S}$, or after being ejected from a neighborhood thereof, as described by Lemma 3.3. For the definition of the functional I see (2.5).

Lemma 3.4. *For any $\delta > 0$, there exist $\varepsilon_0(\delta), \kappa_0, \kappa_1(\delta), \kappa_2(\delta) > 0$ such that the following hold: (I) For any $u \in \mathcal{H}_1$ satisfying $J(u) < J(Q) + \varepsilon_0(\delta)^2$, and $d_Q(u) \geq \delta$, we have*

$$K(u) \leq -\kappa_1(\delta), \quad \textit{or} \quad K(u) \geq \min(\kappa_1(\delta), \kappa_0\|\nabla u\|_2^2). \tag{3.38}$$

(II) For any $u \in \mathcal{H}$ satisfying $I(u) < J(Q) - \delta$, we have

$$K(u) \geq \min(\kappa_2(\delta), \kappa_0\|\nabla u\|_2^2). \tag{3.39}$$

Proof. Part (I) is proved in the same way as [47, Lemma 4.3]. In fact, the situation here is simpler because in contrast to [47] d_Q does not contain the time derivative of the solution and we are dealing with only one K functional. Part (II) is proved via an analogous argument. First, since $M(u) \leq I(u)$ is bounded, the Gagliardo-Nirenberg inequality

$$\|u\|_4^4 \lesssim \|\nabla u\|_2^3\|u\|_2 \tag{3.40}$$

implies that if $\|\nabla u\|_2 \ll 1$ then $K(u) \simeq \|\nabla u\|_2^2$. Hence we may assume that $\|\nabla u\|_2 \gtrsim 1$. Suppose towards a contradiction that $u_n \in \mathcal{H}$ satisfy $I(u_n) < J(Q) - \delta$, $\|\nabla u_n\|_2 \gtrsim 1$ and $K(u_n) \to 0$. Since both $I(u_n)$ and $K(u_n)$ are bounded, the sequence $\{u_n\}$ is bounded in H^1. Hence by extraction of a subsequence, we may assume that $u_n \to u_\infty$ weakly in H^1 and strongly in L^4. Then $I(u_\infty) \leq J(Q) - \delta$ and $K(u_\infty) \leq 0$, so Lemma 2.1 implies that $u_\infty = 0$, hence $\|u_n\|_4 \to 0$, which contradicts that $\|\nabla u_n\|_2 \gtrsim 1$ and $K(u_n) \to 0$. $\square$

Combining the above lemmas, we can now define the sign function $\mathfrak{S}$ which determines the fate of solutions passing by $\mathcal{S}$. The proof is the same as for the analogous statements [47, Lemmas 4.4 and 4.5].

Lemma 3.5. *Let $\delta_S := \delta_X/(2C_*) > 0$ where δ_X and $C_* \ge 1$ are the constants from Lemma 3.3. Let $0 < \delta \le \delta_S$ and*

$$\mathcal{H}_{(\delta)} := \{u \in \mathcal{H}_1 \mid J(u) < J(Q) + \min(d_Q(u)^2/2, \varepsilon_0(\delta)^2)\}, \tag{3.41}$$

where $\varepsilon_0(\delta)$ is given by Lemma 3.4. Then there exists a unique continuous function $\mathfrak{S} : \mathcal{H}_{(\delta)} \to \{\pm 1\}$ satisfying

$$\begin{cases} u \in \mathcal{H}_{(\delta)},\ d_Q(u) \le \delta_E & \implies \mathfrak{S}(u) = -\mathrm{sign}\lambda_1, \\ u \in \mathcal{H}_{(\delta)},\ d_Q(u) \ge \delta & \implies \mathfrak{S}(u) = \mathrm{sign}K(u), \end{cases} \tag{3.42}$$

where we set $\mathrm{sign}0 = +1$*. In addition, we have*

$$\sup\{\|u\|_{H^1} \mid u \in \mathcal{H}_{(\delta_S)},\ \mathfrak{S}(u) = +1\} \lesssim 1. \tag{3.43}$$

4. Virial argument and the one-pass theorem

In this section we establish the following one-pass theorem by means of a suitable virial argument.

Theorem 4.1. *There exist $0 < \varepsilon_* \ll R_* \ll 1$ with the following property: let $u \in C([0,T);\mathcal{H})$ be a forward maximal solution of (1.1) satisfying $M(u) = M(Q)$, $J(u) < J(Q) + \varepsilon^2$ and $d_Q(u(0)) < R$ for some $\varepsilon \in (0, \varepsilon_*]$ and $R \in (2\varepsilon, R_*]$. Then one has the following dichotomy: either $T = \infty$ and $d_Q(u(t)) < R + R^2$ for all $t \ge 0$, or $d_Q(u(t)) \ge R + R^2$ on $t_* \le t < T$ for some finite $t_* > 0$. In the latter case, $\mathfrak{S}(u(t)) \in \{\pm 1\}$ does not change on $t_* \le t < T$; if it is -1, then $T < \infty$, whereas if it is $+1$, then $T = \infty$.*

In the Klein-Gordon case, we were able to use the same R for the dichotomy because the distance function was strictly convex in t. For NLS it may exhibit oscillations on the order of $O(R^3)$, and so we need some room (we chose R^2) to ensure a true ejection from the small neighborhood.

In Section 5 we will show that the solution in fact scatters to zero if $\mathfrak{S}(u(t)) = +1$ for large time. The proof of Theorem 4.1 will take up this entire section. In fact, most work goes into proving the no-return statement, as the finite time blowup vs. global existence dichotomy then follows easily. Indeed, the global existence in the $\mathfrak{S} = +1$ region readily follows from the a priori H^1 bound in Lemma 3.5.

We now turn to the details. We may assume that u does not stay very close to $\mathcal{S}$ for all $t > 0$, so that we can apply the ejection Lemma 3.3 at some time $t_* > 0$. Recall Ogawa-Tsutsumi's *saturated virial identity* [48, (3.5)]

$$\begin{aligned} \partial_t \langle \phi_m u | i u_r \rangle = \int_{\mathbb{R}^3} 2|u_r|^2 \partial_r \phi_m \, dx - |u|^2 \Delta(\partial_r/2 + 1/r)\phi_m \, dx \\ - \int_{\mathbb{R}^3} |u|^4 (\partial_r/2 + 1/r)\phi_m \, dx, \end{aligned} \tag{4.1}$$

where the smooth bounded radial function ϕ_m is chosen as follows:

$$\phi_m(r) = m\phi(r/m), \quad \phi_m(0) = 0 \le \phi_m'(r) \le 1 = \phi_m'(0), \quad \phi_m''(r) \le 0. \tag{4.2}$$

Notice that with this choice of ϕ_m, eq. (4.1) is not merely a cut-off of the virial identity, but rather a "smooth interpolate" of the latter with the Morawetz estimate

for large $|x|$. This is indeed crucial for the following arguments, which are slightly more delicate than those in [48].

The idea (as in [47]) is now to combine the hyperbolic structure of Lemma 3.3 close to $\mathcal{S}$ with the variational structure in Lemma 3.4 away from $\mathcal{S}$, in order to control the virial identity through $K(u)$. We choose $\delta_* > 0$ as the distance threshold between the two regions in $\mathcal{H}_1$: for $d_Q(u) < \delta_*$ we use the hyperbolic estimate in Lemma 3.3, and for $d_Q(u) > \delta_*$ we use the variational estimate in Lemma 3.4. Hence $\delta_*, \varepsilon_*, R_*$ should satisfy

$$\varepsilon_* \ll R_* \ll \delta_* \ll \delta_S, \quad \varepsilon_* \leq \varepsilon_0(\delta_*). \tag{4.3}$$

Below, we shall impose further smallness conditions on $\delta_*, R_*, \varepsilon_*$. Afterward, R_* and then ε_* need to be made even smaller in order to satisfy the above conditions, depending on δ_*.

Suppose towards a contradiction that u solves the NLS equation (1.1) on $[0, T)$ in $\mathcal{H}_1$ satisfying for some $0 < T_1 < T_2 < T_3 < T$ and all $t \in (T_1, T_3)$,

$$d_Q(u(0)) < R = d_Q(u(T_1)) = d_Q(u(T_3)) < d_Q(u(t)), \quad d_Q(u(T_2)) \geq R + R^2, \tag{4.4}$$

as well as $J(u) < J(Q) + \varepsilon^2$, for some $\varepsilon \in (0, \varepsilon_*]$ and $R \in (2\varepsilon, R_*]$.

Lemma 3.5 implies that $\mathfrak{s} := \mathfrak{S}(u(t)) \in \{\pm 1\}$ is well-defined and constant on $T_1 \leq t \leq T_3$.

We apply the ejection Lemma 3.3 first from $t = T_1$ forward in time. Then by the lemma, there exists $T_1' \in (T_1, T_1 + T_* R)$ such that $d_Q(u(t))$ increases for $t > T_1'$ until it reaches δ_X, and $d_Q(u(T_1')) = R + O(R^3) < d_Q(u(T_2)) \ll \delta_X$. Hence $T_1 < T_1' < T_2$, and by the lemma there is $T_1'' \in (T_1', T_3)$ such that $d_Q(u(t))$ increases exponentially on (T_1', T_1''), $d_Q(u(T_1'')) = \delta_X$ and on (T_1, T_1''),

$$d_Q(u(t)) \simeq e^{\mu(t - T_1)} R, \quad \mathfrak{s} K(u(t)) \gtrsim (e^{\mu(t-T_1)} - C_*) R. \tag{4.5}$$

We can argue in the same way from $t = T_3$ backward in time to obtain a time interval $(T_3'', T_3) \subset (T_1'', T_3)$, so that $d_Q(u(T_3'')) = \delta_X$,

$$d_Q(u(t)) \simeq e^{\mu(T_3 - t)} R, \quad \mathfrak{s} K(u(t)) \gtrsim (e^{\mu(T_3 - t)} - C_*) R \quad (T_3'' < t < T_3), \tag{4.6}$$

and $d_Q(u(t))$ is decreasing in the region $d_Q(u(t)) \geq R + R^2$. Moreover, from any $\tau \in (T_1'', T_3'')$ where $d_Q(u(\tau)) < \delta_*$ is a local minimum, we can apply the ejection lemma both forward and backward in time, thereby obtaining an open interval $I_\tau \subset (T_1'', T_3'')$ so that $d_Q(u(\partial I_\tau)) = \{\delta_X\}$,

$$d_Q(u(t)) \simeq e^{\mu|t-\tau|} d_Q(u(\tau)), \quad \mathfrak{s} K(u(t)) \gtrsim (e^{\mu|t-\tau|} - C_*) d_Q(u(\tau)) \quad (t \in I_\tau), \tag{4.7}$$

and $d_Q(u(t))$ is monotone in the region $d_Q(u(t)) \geq 2 d_Q(u(\tau))$, which is the reason for $I_\tau \subset (T_1'', T_3'')$. Moreover, the monotonicity away from τ implies that any two intervals I_{τ_1} and I_{τ_2} for distinct minimal points τ_1 and τ_2 are either disjoint or identical. Therefore, we have obtained disjoint open subintervals $I_1, \dots, I_n \subset (T_1, T_3)$ with $n \geq 2$, where we have either (4.5), (4.6), or (4.7) with $\tau = \tau_j \in I_j$, and at the remaining times

$$t \in I' := (T_1, T_3) \setminus \bigcup_{j=1}^{n} I_j, \tag{4.8}$$

we have $d_Q(u(t)) \ge \delta_*$, so that we can apply Lemma 3.4 to obtain

$$\begin{cases} K(u(t)) \ge \min(\kappa_1(\delta_*), \kappa_2\|\nabla u(t)\|_2^2) & (\mathfrak{s} = +1), \\ K(u(t)) \le -\kappa_1(\delta_*) & (\mathfrak{s} = -1). \end{cases} \quad (t \in I') \tag{4.9}$$

4.1. **Virial estimate in the blow-up case $\mathfrak{s} = -1$.** In this case, we choose ϕ just as in [48]:

$$\phi(r) = \begin{cases} r & (r \le 1), \\ \frac{3}{2} & (r \ge 2), \end{cases} \tag{4.10}$$

and then rewrite (4.1) in the form

$$\partial_t \langle \phi_m u | i u_r \rangle = 2K(u) - 2\int |u_r|^2 f_{0,m}\, dx + \int [|u|^2 f_{1,m}/r^2 + |u|^4 f_{2,m}]\, dx, \tag{4.11}$$

where $f_{j,m} = f_j(r/m)$ are smooth functions supported on $r > m$, defined by

$$f_0 = 1 - \phi_r, \quad f_1 = -r^2\Delta(\partial_r/2 + 1/r)\phi, \quad f_2 = 3/2 - (\partial_r/2 + 1/r)\phi. \tag{4.12}$$

For the L^4 error term we use the radial Sobolev inequality as in [48]

$$\|u\|_{L^4(r>m)}^4 \lesssim m^{-2}\|u\|_{L^2(r>m)}^3 \|u_r\|_{L^2(r>m)}, \tag{4.13}$$

see (C.8). In order to absorb the kinetic term, noting that $f_0' \ge 0$ and $|f_2| \lesssim f_0$, we use the weighted version of the above inequality:

$$\begin{aligned} &\int_m^\infty f_{0,m}(r)|u|^4(r) r^2\, dr = \int_m^\infty \int_s^\infty f_{0,m}'(s)|u|^4(r) r^2\, dr ds \\ &\lesssim \int_m^\infty f_{0,m}'(s) s^{-2}\|u\|_{L^2(r>s)}^3 \|u_r\|_{L^2(r>s)}\, ds \\ &\lesssim \int_m^\infty f_{0,m}'(s) \int_s^\infty \lambda |u_r|^2(r) r^2\, dr ds + \int_m^\infty f_{0,m}'(s)\lambda^{-1} s^{-4}\|u\|_{L^2(r>s)}^6\, ds \\ &\le \lambda \int_m^\infty f_{0,m}(r)|u_r|^2(r) r^2\, dr + \lambda^{-1} m^{-4}\|u\|_{L^2(r>m)}^6, \end{aligned} \tag{4.14}$$

which holds uniformly for $\lambda > 0$. Choosing $\lambda > 0$ small (in terms of the constants in those Sobolev inequalities), we obtain

$$\begin{aligned} \partial_t \langle \phi_m u | i u_r \rangle &\le 2K(u) + O(m^{-4}\|u\|_{L^2(r>m)}^6) + O(m^{-2}\|u\|_{L^2(r>m)}^2) \\ &\le 2K(u) + O(m^{-2}). \end{aligned} \tag{4.15}$$

We can now prove Theorem 4.1 in the blow-up case $\mathfrak{s} = -1$, by integrating (4.15), combined with (4.5)–(4.9). We thus obtain

$$\begin{aligned} -[\langle \phi_m u | i u_r \rangle]_{T_1}^{T_3} &\gtrsim \sum_{j=1}^n \int_{I_j} [(e^{\mu|t-\tau_j|} - C_*) d_Q(u(\tau_j)) - Cm^{-2}]\, dt \\ &\quad + \int_{I'} [\kappa_1(\delta_*) - Cm^{-2}]\, dt \\ &\gtrsim n\delta_X \ge \delta_X, \end{aligned} \tag{4.16}$$

provided that

$$m^{-2} \lesssim R_*, \quad m^{-2} \ll \kappa_1(\delta_*). \tag{4.17}$$

On the other hand, since $d_Q(u(t)) = R$ at $t = T_1, T_3$ and since Q is exponentially decaying,

$$\left|[\langle \phi_m u | i u_r \rangle]_{T_1}^{T_3}\right| \lesssim R + mR^2 \lesssim R_* \ll \delta_X, \tag{4.18}$$

if we choose $m = 1/R$. Comparing this bound with (4.16) leads to a contradiction. In conclusion, the solution $u(t)$ cannot return to the R-ball from the $\mathfrak{s} = -1$ side if we choose $\delta_*, R_*, \varepsilon_* > 0$ such that

$$R_*^2 \ll \kappa_1(\delta_*) \tag{4.19}$$

and (4.3) are satisfied. Therefore, if u extends to $t \to +\infty$, then $T_3 = \infty$ and so (4.16) with $m = 1/R$ fixed implies

$$m\|u_r(t)\|_2 \gtrsim -\langle \phi_m u | i u_r \rangle \to \infty \tag{4.20}$$

as $t \to \infty$. Hence for large $t \gg 1$ we have $K(u(t)) = 3E(u(t)) - \frac{1}{2}\|\nabla u(t)\|_2^2 \to -\infty$. Thus for $T_1 \ll \forall t_1 < \forall t_2$,

$$[\langle \phi_m u | i u_r \rangle]_{t_1}^{t_2} \lesssim -\int_{t_1}^{t_2} \|u_r(t)\|_2^2 \, dt, \tag{4.21}$$

and so

$$m\|u_r(t_2)\|_{L^2_x} \gtrsim -C\, m\|u_r(t_1)\|_{L^2_x} + \int_{t_1}^{t_2} \|u_r(t)\|_2^2 dt, \tag{4.22}$$

which leads to blow-up of $\|u_r(t)\|_2$ in finite time from the blowup exhibited by $f'(t) \gtrsim f^2(t)$. This concludes the proof of Theorem 4.1 in the case $\mathfrak{s} = -1$.

4.2. **Virial estimate in the scattering case $\mathfrak{s} = +1$.** In this case, the sign-definiteness in the variational region becomes more delicate. For simplicity, we make a specific choice[2] for ϕ:

$$\phi(r) = \frac{r}{1+r}, \tag{4.23}$$

and rewrite (4.1) in a different way:

$$\partial_t \langle \phi_m u | i u_r \rangle = 2K(\chi_m u) + \int \Big[\frac{|u|^2}{m^2} f_{3,m} + |u|^4 f_{4,m}\Big] \, dx, \tag{4.24}$$

where $\chi_m(r) = \chi(r/m)$ and $f_{j,m}(r) = f_j(r/m)$ are smooth functions defined by

$$\begin{aligned} &\chi(r) = \sqrt{\phi_r}(r) = \frac{1}{1+r}, \quad f_3(r) := -2|\chi_r(r)|^2 + \frac{\phi_{rrr}(r)}{2} = \frac{1}{(1+r)^4}, \\ &- f_4(r) := -\Big[\frac{3(\phi_r)^2}{2} - \frac{\phi_r}{2} - \frac{\phi}{r}\Big](r) = \frac{r(2r^2+7r+8)}{2(1+r)^4} \simeq \frac{\phi(r)^2}{r} \end{aligned} \tag{4.25}$$

In order to absorb the $|u|^4$ term, we use another radial Sobolev inequality (C.6)

$$\int_m^\infty |u|^4 r dr \lesssim \int_m^\infty |u|^2 r^2 dr \int_m^\infty |u_r|^2 dr. \tag{4.26}$$

[2]The important property of ϕ in the $\mathfrak{s} = +1$ case is that the convergence as $r \to \infty$ is slow.

The same argument as in (4.14) transforms it into

$$\int_0^\infty |u|^4\phi_m^2 r dr \lesssim \|u\|_{L^2}^2 \int_0^\infty |u_r|^2\phi_m^2\, dr. \tag{4.27}$$

Since $(\chi_m')^2 \simeq f_{3,m}/m^2$, the right-hand side is estimated by

$$\int_0^\infty |u_r|^2\phi_m^2 dr \lesssim \int_0^\infty (|(\chi_m u)_r|^2 + m^{-2}|u|^2 f_{3,m})r^2\, dr. \tag{4.28}$$

Thus we obtain

$$\int |u|^4 f_{4,m}\, dx \lesssim m^{-1}\left[\|\nabla(\chi_m u)\|_{L^2}^2 + \int \frac{|u|^2}{m^2} f_{3,m}\, dx\right]. \tag{4.29}$$

In particular it is $O(m^{-1})$ since u is bounded in H^1 by Lemma 3.5.

In the hyperbolic region, the cut-off in K has little impact, since by the same expansion as in (3.37), we have

$$\begin{aligned} K(\chi_m u) &= K(Q + \chi_m w + (\chi_m - 1)Q) \\ &= -\lambda_1\mu\langle Q|\psi\rangle - \langle 2Q + Q^3|\chi_m\gamma + (\chi_m - 1)(Q + 2\lambda_1\varphi)\rangle \\ &\qquad + O(\|w\|_{H^1}^2 + \|(\chi_m - 1)Q\|_{H^1}^2) \\ &= -\lambda_1\mu\langle Q|\psi\rangle - \langle 2Q + Q^3|\chi_m\gamma\rangle + O(\|w\|_{H^1}^2 + e^{-m}), \end{aligned} \tag{4.30}$$

thanks to the exponential decay of Q.

In the variational region $d_Q(u) > \delta_*$, if $\|\nabla u\|_2 \le \mu$ for some small $\mu > 0$, then by Gagliardo-Nirenberg (3.40), we have

$$K(\chi_m u) \simeq \|\nabla\chi_m u\|_2^2. \tag{4.31}$$

Otherwise $\|\nabla u\|_2 > \mu$ and so we have from (4.9),

$$K(u(t)) \ge \kappa_3(\delta_*) := \min(\kappa_1(\delta_*), \mu) \quad (t \in I'). \tag{4.32}$$

Hence, if we choose $\varepsilon_* > 0$ so that

$$\varepsilon_*^2 < \kappa_3(\delta_*)/6, \tag{4.33}$$

then $d_Q(u) > \delta_*$ and $J(u) < J(Q) + \varepsilon^2$ imply

$$I(u) < J(Q) - \kappa_3(\delta_*)/3. \tag{4.34}$$

Since $I(\chi_m u) \le I(u)$, Lemma 3.4 (II) yields

$$K(\chi_m u) \ge \min(\kappa_4(\delta_*), \kappa_0\|\nabla\chi_m u\|_2^2), \tag{4.35}$$

where $\kappa_4(\delta_*) := \kappa_2(\kappa_3(\delta_*)/3)$. This bound is valid including the case $\|\nabla u\|_2 \le \mu$ (making κ_0 smaller if necessary).

Now choose $m > 1$ so large that we have

$$m^{-1} \ll \min(\kappa_4(\delta_*), \kappa_0) =: \kappa_5(\delta_*). \tag{4.36}$$

Then (4.29) and (4.35) imply

$$2K(\chi_m u) + \int \left[\frac{|u|^2}{m^2} f_{3,m} + |u|^4 f_{4,m}\right] dx \ge 0. \tag{4.37}$$

Hence we obtain the monotonicity

$$\partial_t \langle \phi_m u | i u_r \rangle \geq 0, \tag{4.38}$$

in the variational region $d_Q(u) \geq \delta_*$, i.e., for $t \in I'$, cf. (4.9). By the same argument as in the case of $\mathfrak{s} = -1$, we now arrive at a contradiction by choosing $m \geq 1/R$ and $R_*, \delta_*, \varepsilon_* > 0$ so that (4.3), (4.33) and

$$R_*^2 \ll \kappa_5(\delta_*) \tag{4.39}$$

are satisfied. This concludes the proof of Theorem 4.1.

5. Scattering for $K > 0$ solutions

In this section we establish the following scattering result, following the proof scheme of [35]. Let R_* be a fixed choice of R as in Theorem 4.1.

Proposition 5.1. *Let $\varepsilon_*, R_* > 0$ be as in Theorem 4.1. There exists $\mathcal{N} < \infty$ such that if a solution u of the NLS equation (1.1) on $[0, \infty)$ satisfies $M(u) = M(Q)$ and $E(u) \leq E(Q) + \varepsilon_*^2$, as well as $d_Q(u(t)) \geq R_*$ and $\mathfrak{S}(u(t)) = +1$ for all $t \geq 0$, then u scatters to 0 as $t \to \infty$ and $\|u\|_{L^4_t((0,\infty);L^4_x)} \leq \mathcal{N}$.*

Proof. Let $\mathcal{U}$ be the collection of all solutions u of (1.1) on $[0, \infty)$ satisfying

$$\begin{aligned} &M(u) = M(Q), \quad E(u) \leq E(Q) + \varepsilon_*^2, \quad d_Q(u[0,\infty)) \subset [R_*, \infty), \\ &\mathfrak{S}(u(t)) = +1 \quad (t \geq 0). \end{aligned} \tag{5.1}$$

Indeed, Lemma 3.5 implies that $\mathfrak{S}(u(t)) = +1$ is preserved as long as $d_Q(u(t)) \geq R_* > 2\varepsilon$, and $u(t)$ is a forward global solution uniformly bounded in H^1. Moreover, the lower bound on d_Q implies, by Lemma 3.2 that

$$\inf_{t \geq 0} \operatorname{dist}_{L^2}(u(t), \mathcal{S}_1) \gtrsim \min(R_*, \delta_E^2) = R_*. \tag{5.2}$$

For each $E > 0$, let $\mathcal{N}(E)$ be defined as

$$\mathcal{N}(E) := \sup\{\|u\|_{L^4_t((0,\infty);L^4_x)} \mid u \in \mathcal{U},\ E(u) \leq E\} \tag{5.3}$$

See Section A for the relevance of L^4_{tx}. We know by [31] that $\mathcal{N}(E) < \infty$ for $E < E(Q)$. In order to extend this property to $E(Q) + \varepsilon_*^2$, put

$$E^\star = \sup\{E > 0 \mid \mathcal{N}(E) < \infty\} \tag{5.4}$$

Then $E^\star \geq E(Q)$ and assume towards a contradiction that

$$E^\star = E(Q) + \varepsilon^2 < E(Q) + \varepsilon_*^2, \tag{5.5}$$

where $0 \leq \varepsilon < \varepsilon_*$. We consider a nonlinear profile decomposition in the sense of Bahouri-Gérard [1] for any sequence $u_n \in \mathcal{U}$ satisfying

$$E(u_n) \to E^\star, \quad \|u_n\|_{L^4_t((0,\infty);L^4_x)} \to \infty. \tag{5.6}$$

We are going to show that the remainder in the decomposition vanishes in a suitable sense and that there is only one profile, which is a critical element, i.e.,

$$u_\star \in \mathcal{U}, \quad E(u_\star) = E^\star, \quad \|u_\star\|_{L^4_t((0,\infty);L^4_x)} = \infty. \tag{5.7}$$

Before starting the decomposition for u_n, we translate u_n in t to achieve

$$d_Q(u_n(0)) \geq \delta_X, \quad K(u_n(0)) > 6\varepsilon_*^2. \tag{5.8}$$

Since $d_Q(u_n(t))$ will never come down to R_*, the ejection Lemma 3.3 gives $0 < T_n \lesssim \log(\delta_X/R_*)$ so that $d_Q(u_n(T_n)) \geq \delta_X$. Since $\mathfrak{S} = +1$, we have a uniform H^1 bound on u_n and so, in view of Lemma A.1 together with $\|u_n\|_{L^4_{t,x}(0,\infty)} \to \infty$, we have $\|\nabla u_n(T_n)\|_2 > \mu$. Hence by Lemma 3.4,

$$K(u_n(T_n)) \geq \min(\kappa_1(\delta_X), \kappa_0\mu^2) \geq \kappa_3(\delta_*) > 6\varepsilon_*^2, \tag{5.9}$$

where κ_3 is defined in (4.32) and we used the condition (4.33) on ε_*. Translating $u_n := u_n(t - T_n)$, we may assume (5.8).

Now apply Proposition A.2 to the sequence $\{u_n(0)\}$. This yields, cf. (A.2)

$$e^{-it\Delta}u_n(0) = \sum_{0\leq j<k} e^{-i(t+t_n^j)\Delta}v^j + \gamma_n^k(t) \tag{5.10}$$

We take here k sufficiently large so that γ_n^k is small in the sense of (A.3); it will always be assumed that n is large. The first step consists in showing that due to (5.5) one has $v^j = 0$ for all but one j as well as

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma_n^k\|_{L_t^\infty H^1} = 0 \tag{5.11}$$

By the partition property (A.4) one has

$$M(u_n) = \sum_{0\leq j<k} M(v^j) + M(\gamma_n^k) + o(1) \tag{5.12}$$

as $n \to \infty$. Assume that $M(v^0) > 0$ and $M(v^1) > 0$. Then $M(v^j) < M(Q)$ for each j. From (5.8), $J(u_n) < J(Q) + \varepsilon_*^2$, and (A.4), we infer that

$$\begin{aligned} J(Q) - \varepsilon_*^2 &> \limsup_{n\to\infty}[J(u_n) - K(u_n(0))/3] \\ &\geq \limsup_{n\to\infty} G(u_n(0)) = \sum_{j<k} G(v^j) + \limsup_{n\to\infty} G(\gamma_n^j), \end{aligned}$$

where we used that $G(\varphi) \simeq \|\varphi\|_{H^1}^2$ is conserved by the linear flow. Since G is positive definite, we obtain $G(v^j) < J(Q) - \varepsilon_*^2$ for all j, which implies, via the minimizing property (2.6) and the invariance of G under the linear flow, that $K(e^{-it\Delta}v^j) \geq 0$ for all $t \in \mathbb{R}$. Let $t_n^j \to t_\infty^j \in [-\infty, \infty]$ and let u^j be the nonlinear profile associated with v^j, i.e., that solution of (1.1) satisfying

$$\lim_{t\to t_\infty^j} \|u^j(t) - e^{-it\Delta}v^j\|_{H^1} = 0, \tag{5.13}$$

which exists at least locally around $t = t_\infty^j$, either by solving the Cauchy problem at $t_\infty^j \in \mathbb{R}$ or by applying the wave operator at $t_\infty^j \in \{\pm\infty\}$. By the preceding,

$$M(u^j) = M(v^j) < M(Q), \quad K(u^j(t)) \geq 0 \quad \forall\, t \text{ near } t_\infty^j \tag{5.14}$$

In particular, $E(u^j) \geq 0$. We have the following partition of the nonlinear energy

$$E(u_n) = \sum_{0\leq j<k} E(u^j) + E(\gamma_n^k(0)) + o(1) \tag{5.15}$$

as $n \to \infty$. Since

$$\|u\|_4^4 \lesssim \|\nabla u\|_2^2 \|u\|_3^2$$

and (A.3) imply that $E(\gamma_n^k) \simeq \|\gamma_n^k\|_{H^1}^2$ provided k is large, we conclude that

$$0 \le E(u^j) \le \lim_{n\to\infty} E(u_n) = E(Q) + \varepsilon^2 \quad \forall\, j. \tag{5.16}$$

If $E(u^j) < E(Q)$ for all j, then we conclude by the preceding and [31] that each u^j exists globally and scatters with $\|u^j\|_{S(\mathbb{R})} < \infty$, where

$$S(I) := L_t^\infty H_x^1(I) \cap L_t^2 W_x^{1,6}(I) \tag{5.17}$$

is the norm of the Strichartz spaces. But then one has the following *nonlinear profile decomposition*,

$$u_n = \sum_{j<k} u_n^j + \gamma_n^k + \mathrm{err}_n^k, \quad u_n^j := u^j(t + t_n^j) \tag{5.18}$$

where γ_n^k are as in (5.10), and the errors err_n^k satisfy

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\mathrm{err}_n^k\|_{S_0(\mathbb{R})} = 0 \tag{5.19}$$

where

$$S_0(I) = L_t^\infty L_x^2(I) \cap L_{t,x}^4(I) \tag{5.20}$$

is a subcritical Strichartz norm. To see this, first let w_n^j be the nonlinear solution with the initial condition $w_n^j(0) = e^{-it_n^j\Delta} v^j$. By the local theory

$$w_n^j(t) - u^j(t + t_n^j) \to 0 \quad (n \to \infty)$$

in $S(I)$ for some interval $I \ni 0$. Now apply Lemma A.3 with

$$t_0 = 0, \quad u = u_n, \quad w = \sum_{j<k} u_n^j + \gamma_n^k$$

to conclude (5.18). In order to apply this lemma, one needs to verify that the nonlinear interactions between all $\{u^0, u^1, \dots, \gamma_n^k\}$, as well as $|\gamma_n^k|^2\gamma_n^k$, vanish in the $L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}$-norm in the limits as $n \to \infty$ and then $k \to \infty$. However, as usual, this follows by expanding the cubic nonlinearity and using the Strichartz control available for each of these functions. Since (5.18) contradicts (5.6), we must have that at least one u^j, and therefore exactly one, say u^0, satisfies $E(u^0) \ge E(Q)$. But then $E(u^j) \lesssim \varepsilon^2 \ll E(Q)$ for all $j \ge 1$ and these u^j are globally defined and scatter. If u^0 also scatters, then by the same reasoning we get a contradiction. Hence u^0 does not scatter either for $t \to \infty$ or for $t \to -\infty$.

Let $\tilde{u}^0(t,x) = \alpha u^0(\alpha^2 t, \alpha x)$ be the rescaling of u^0 such that $M(\tilde{u}^0) = M(Q)$. Then

$$E(\tilde{u}^0) \le \frac{M(u^0)}{M(Q)}(E(Q) + \varepsilon^2), \tag{5.21}$$

so we have $M(u^0) > M(Q) - O(\varepsilon^2)$ for otherwise one has global scattering for u^0 from [31] and therefore a contradiction via (5.18). Hence $\alpha = 1 + O(\varepsilon^2)$. In view of (5.12) and (5.15), $J(u^j) \lesssim \varepsilon^2$ for all $j \ge 1$. Since

$$J(u^j) \ge G(u^j(t)) \ge \|u^j(t)\|_{H^1}^2/6$$

for all t and any $j \geq 1$, we conclude that $\|u^j\|_{L^\infty H^1} \lesssim \varepsilon$ for each $j \geq 1$. In fact, the same logic with the asymptotic orthogonality yields the following stronger bound

$$\sum_{j\geq 1} \|u^j\|^2_{L^\infty_t H^1_x} + \sup_{k\geq 1} \limsup_{n\to\infty} \|\gamma^k_n\|^2_{L^\infty_t H^1_x} \lesssim \varepsilon^2, \tag{5.22}$$

which will be crucial later.

If $t^0_\infty = \infty$, then by the definition u^0 scatters as $t \to \infty$, and so one obtains a contradiction as before via (5.18).

If $t^0_\infty = -\infty$, then $\tilde{u}^0$ scatters as $t \to -\infty$, and we face the following dichotomy: either $\tilde{u}^0$ satisfies $d_Q(\tilde{u}^0(t)) > 2\varepsilon_*$ on its entire interval of existence, or there exists some time t_* in the interval of existence of u^0 at which

$$d_Q(\tilde{u}^0(\alpha^{-2}t_*)) \leq 2\varepsilon_*. \tag{5.23}$$

In the former case, we infer that $\mathfrak{S}(\tilde{u}^0(t)) = +1$ is preserved from $t = -\infty$, therefore $\tilde{u}^0$ exists globally and satisfies $d_Q(\tilde{u}^0(t)) > R_*$ for large $t > T'$; for this last statement one invokes Lemma 3.3 to obtain ejection, if needed. Then $\tilde{u}^0(t - T')$ is a critical element, since $\tilde{u}^0$ does not scatter for $t \to \infty$. In the latter case (5.23), we have

$$\|u^0\|_{L^4((-\infty,t_*+C];L^4_x)} < \infty \text{ with } C \gtrsim 1,$$

and thus also (5.18) on the time interval $(-\infty, t_* - t^0_n)$. But then, by (5.22),

$$\mathrm{dist}_{L^2}(u_n(t_* - t^0_n), \mathcal{S}_1) \leq d_Q(\tilde{u}^0(\alpha^{-2}t_*)) + O(\varepsilon) \lesssim \varepsilon_* \ll R_*,$$

and $t_* - t^0_n \to \infty$, contrary to (5.2) for u_n.

The only remaining case is $t^0_\infty \in \mathbb{R}$. In that case we use the general fact that the nonlinear profile decomposition (5.18) holds locally around $t = 0$. In addition to that, we have by (5.8) and (5.22),

$$d_Q(\tilde{u}^0(\alpha^{-2}t^0_\infty)) \geq \limsup_{n\to\infty} d_Q(u_n(0)) - O(\varepsilon) > \delta_S, \tag{5.24}$$

and so $\mathfrak{S}(\tilde{u}^0(\alpha^{-2}t^0_\infty)) = +1$ since $K(\tilde{u}^0(\alpha^{-2}t^0_\infty)) = \alpha K(u^0(t^0_\infty)) \geq 0$. If u^0 scatters as $t \to \infty$, then we obtain a contradiction as before via (5.18). Hence $\tilde{u}^0$ does not scatter for $t \to \infty$. If $d_Q(\alpha^{-2}\tilde{u}^0(t)) > 2\varepsilon_*$ for all $t \geq t^0_\infty$ on the maximal interval of existence of $\tilde{u}^0$, then $\mathfrak{S}(\tilde{u}^0) = +1$ for these t, which implies that $\tilde{u}^0$ is forward global and a critical element after time translation (use the ejection Lemma 3.3 if needed to conclude that $d_Q(\tilde{u}^0(t)) \geq R_*$ for large t). Otherwise, there exists $t_* > t^0_\infty$ minimal so that $d_Q(\tilde{u}^0(\alpha^{-2}t_*) < 2\varepsilon_*$. But then (5.18) remains valid near t_* and one again obtains a contradiction to

$$\mathrm{dist}_{L^2}(u_n(t_* - t^0_\infty), \mathcal{S}_1) \gtrsim R_* \gg \varepsilon_*.$$

Therefore, the conclusion is that u^0 is a critical element after time translation, $E(u^0) = E^\star$, $M(u^0) = M(Q)$, and so $v^j = 0$ for all $j \geq 1$ as well as $\gamma^k_n \to 0$ in the energy sense as $n \to \infty$. Hence (after extracting a subsequence)

$$\lim_{n\to\infty} \|u_n(T_n) - u^0(t^0_n)\|_{H^1} = 0, \tag{5.25}$$

where T_n is the time shift for (5.8). Both T_n and t^0_n are bounded above for $n \to \infty$. If $t^0_n \to -\infty$ then u^0 scatters as $t \to -\infty$, and the local theory of the wave operator implies that $\|u_n\|_{L^4_{t,x}(-\infty,T_n)} \to 0$.

Applying the above result to the sequence $u_n := u^0(t+\tau_n)$ for arbitrary $\tau_n \to \infty$, one now concludes that the forward trajectory $\{u^0(t)\}_{t\geq 0}$ is precompact in H^1, since $t_n^0 \to -\infty$ would imply that $\|u^0\|_{L^4_{t,x}(-\infty,\tau_n)} \to 0$ which is a contradiction.

Finally, integrating the saturated virial identity (4.24) from Section 4.2 between some positive time and $t=\infty$ now proves that such a critical element cannot exist. Note that $K(\chi_m u)$ has a positive lower bound in the variational region (for large m) by the precompactness of the forward trajectory of the critical element. This shows that $E^\star < E(Q)+\varepsilon_*^2$ is impossible, concluding the proof. □

6. The proof of Theorem 1.1

Let $\mathcal{H}^\varepsilon$ and $\mathcal{H}^\varepsilon_\alpha$ be as defined in (1.3), (1.4), respectively. We introduce the following subsets according to the global behavior of the solution $u(t)$ of (1.1): for $\sigma=\pm$ respectively,

$$\begin{aligned} \mathcal{S}^\varepsilon_\sigma &= \{u(0)\in\mathcal{H}^\varepsilon_1 \mid u(t) \text{ scatters as } \sigma t\to\infty\}, \\ \mathcal{T}^\varepsilon_\sigma &= \{u(0)\in\mathcal{H}^\varepsilon_1 \mid u(t) \text{ trapped by } \mathcal{S}_1 \text{ for } \sigma t\to\infty\}, \\ \mathcal{B}^\varepsilon_\sigma &= \{u(0)\in\mathcal{H}^\varepsilon_1 \mid u(t) \text{ blows up in } \sigma t>0\}. \end{aligned} \tag{6.1}$$

The trapping for $\mathcal{T}^\varepsilon_+$ can be characterized as follows, with any $R\in(2\varepsilon,R_*)$:

$$\exists T>0, \quad \forall t>T, \quad d_Q(u(t))<R.$$

Obviously those sets are increasing in ε, and have the conjugation property

$$X^\varepsilon_\mp = \{\overline{u}(0)\in\mathcal{H}^\varepsilon_1 \mid u(0)\in X^\varepsilon_\pm\},$$

for $X=\mathcal{S},\mathcal{T},\mathcal{B}$. Moreover, $\mathcal{S}_+$ and $\mathcal{T}_+$ are forward invariant by the flow of NLS, while $\mathcal{S}_-$ and $\mathcal{T}_-$ are backward invariant. By what we have done so far

$$\mathcal{H}^\varepsilon_1 = \mathcal{S}^\varepsilon_+\cup\mathcal{T}^\varepsilon_+\cup\mathcal{B}^\varepsilon_+ = \mathcal{S}^\varepsilon_-\cup\mathcal{T}^\varepsilon_-\cup\mathcal{B}^\varepsilon_-,$$

with the union being disjoint for each sign. It follows from the scattering theory that $\mathcal{S}^\varepsilon_\pm$ are open (relatively, in $\mathcal{H}^\varepsilon_1$). We claim the same for $\mathcal{B}^\varepsilon_\pm$, which is not a general fact. Thus, suppose that $u(t)$ blows up at $0<T<\infty$. This is equivalent to $\|\nabla u(t)\|_2\to\infty$ as $t\to T-$. Since

$$K(u(t)) = 3E(u) - \frac{1}{2}\|\nabla u(t)\|_2^2$$

it follows that $K(u(t))\to-\infty$ as $t\to T-$. We now claim that if $v(0)\in\mathcal{H}^\varepsilon_1$ with $\|v(0)-u(T')\|_{H^1}<1$ where $T'<T$ is fixed and very close to T, then $v(0)$ leads to a solution $v(t)$ which blows up in finite time. Suppose not. Then from the energy constraint $E(v(0))<E(Q)+\varepsilon^2$ we know that $K(v(t))$ can only change sign by coming very close to $\mathcal{S}_1$. But since (4.15), with $m\gg 1$ fixed, implies that $\langle\phi_m v|iv_r\rangle$ decreases on any time interval where $K(v(t))\le -1$, and since $\langle\phi_m v|iv_r\rangle=0$ on $\mathcal{S}_1$, it follows that $K(v(t))\le-1$ for all $t\ge 0$. But then we obtain a contradiction via (4.15). Thus v blows up in finite time as claimed. Therefore, $\mathcal{B}^\varepsilon_\pm$ are also open, so $\mathcal{T}^\varepsilon_\pm$ are relatively closed in $\mathcal{H}^\varepsilon_1$.

Since $\mathcal{B}^\varepsilon_+$ and $\mathcal{S}^\varepsilon_+$ are disjoint open, they are separated by $\mathcal{T}^\varepsilon_+$, that is, any two points from $\mathcal{S}^\varepsilon_+$ and $\mathcal{B}^\varepsilon_+$ cannot be joined by a curve in $\mathcal{H}^\varepsilon_1$ without passing through $\mathcal{T}^\varepsilon_+$.

In a small ball around $\mathcal{S}_1$, it is easy to see by means of the linearized flow that the open intersections $\mathcal{B}^\varepsilon_\pm \cap \mathcal{S}^\varepsilon_\mp$, $\mathcal{B}^\varepsilon_+ \cap \mathcal{B}^\varepsilon_-$ and $\mathcal{S}^\varepsilon_+ \cap \mathcal{S}^\varepsilon_-$ are all non-empty for any $\varepsilon > 0$. Examples of solutions belonging to the first set are given by data in (3.5)

$$\lambda_+(0) = -\lambda_-(0) = \varepsilon\rho, \quad \gamma(0) = \alpha Q', \tag{6.2}$$

with real parameters, whence

$$\begin{aligned} u(0) &= Q + \varepsilon\rho\mathfrak{G}_+ - \varepsilon\rho\mathfrak{G}_- + \alpha Q' \\ &= Q + \alpha Q' - 2i\varepsilon\rho\psi \end{aligned} \tag{6.3}$$

and thus

$$\begin{aligned} M(u) &= M(Q + \alpha Q') + 4\varepsilon^2\rho^2 M(\psi) \\ &= M(Q) + \alpha\langle Q|Q'\rangle + \alpha^2 M(Q') + 4\varepsilon^2\rho^2 M(\psi) \end{aligned} \tag{6.4}$$

where $|\rho| \ll 1$, and $\alpha > 0$ should be chosen such that $M(u) = M(Q)$. Then the linearized flow of $\lambda_1(t)$ is $\lambda_1^{(0)}(t) = \varepsilon\rho\sinh(\mu t)$ and $\|\gamma^{(0)}\|_2 = O(\varepsilon^2\rho^2)$. In fact, the estimates (3.30) show that the true $\lambda_1(t), \gamma(t)$ deviate from these only by quadratic corrections $O(\varepsilon^2\rho^2)$. Therefore, at exit time from the δ_X-ball one has $t\mathfrak{S}(u(t))$ of a fixed sign. Hence, choosing the sign of ρ correctly leads to solutions $u \in \mathcal{B}^\varepsilon_- \cap \mathcal{S}^\varepsilon_+$, or $\tilde{u} \in \mathcal{B}^\varepsilon_+ \cap \mathcal{S}^\varepsilon_-$, respectively.

The analogous construction with $\pm\cosh(\mu t)$ instead of $\sinh(\mu t)$ furnishes examples of solutions $u_\pm$ belonging to $\mathcal{B}^\varepsilon_+ \cap \mathcal{B}^\varepsilon_-$ and $\mathcal{S}^\varepsilon_+ \cap \mathcal{S}^\varepsilon_-$, respectively. It is clear that these constructions actually give open nonempty sets of solutions relative to $\mathcal{H}^\varepsilon_1$ (indeed, we can perturb $\gamma(0)$ within $O(\varepsilon^2\rho^2)$).

Next, note that by construction $\|u(t) - u_+(t)\|_{H^1} \ll \varepsilon$ while $d_Q(u(t)) \gg \varepsilon$ for some large times (but before exit from the δ_X-ball). It follows that we may connect $u(t)$ with $u_+(t)$ by a curve segment Γ within $\mathcal{H}^\varepsilon_1$ and within the set $\mathfrak{S} < 0$ and $d_Q(u) \gg \varepsilon$. Since $u(t) \in \mathcal{S}^\varepsilon_-$ and $u_+(t) \in \mathcal{B}^\varepsilon_-$, there exists $p_0 \in \mathcal{T}^\varepsilon_- \cap \Gamma$. Since the solution starting from p_0 enters the 3ε-ball around $\mathcal{S}_1$ as $t \to -\infty$, and initially p_0 is much further away and also $\mathfrak{S}(p_0) = -1$, we conclude by the one-pass theorem that $p \in \mathcal{B}^\varepsilon_+$. Hence $\mathcal{T}^\varepsilon_- \cap \mathcal{B}^\varepsilon_+$ is non-empty as well. In the same way, we can find a point on the curve connecting $u(t)$ and $u_-(t)$ for some $t < 0$, which is in $\mathcal{T}^\varepsilon_+ \cap \mathcal{S}^\varepsilon_-$. Therefore, $\mathcal{T}^\varepsilon_\pm \cap \mathcal{B}^\varepsilon_\mp$ and $\mathcal{T}^\varepsilon_\pm \cap \mathcal{S}^\varepsilon_\mp$ are both not empty. Taking the limit $\varepsilon \to +0$, it is easy to observe that they contain infinitely many points on different energy levels.

The sets $\mathcal{T}^\varepsilon_+ \cap \mathcal{T}^\varepsilon_-$ contain all of $\mathcal{S}_1$, and are therefore not empty. By considering curves on the hyperplane $\{\operatorname{Im} u = 0\}$ connecting $u_+(0)$ with $u_-(0)$ (the solutions from before) and which are disjoint from $\mathcal{S}_1$, we obtain infinitely many points in $\mathcal{T}^\varepsilon_+ \cap \mathcal{T}^\varepsilon_- \setminus \mathcal{S}_1$.

By a simple scattering theory argument one can check that $\mathcal{S}^\varepsilon_\pm$ is path-wise connected, as is every slice of fixed energy of this set (all relative to $\mathcal{H}^\varepsilon_1$). See [47] (7.15)–(7.19) for details. Therefore, $\mathcal{S}^\varepsilon_+$ is its own connected component. To find a curve connecting 0 to ∞ in H^1 within that set, follow a solution in $\mathcal{B}^\varepsilon_- \cap \mathcal{S}^\varepsilon_+$ to blow-up time. This concludes the proof of Theorem 1.1.

7. Construction of the center-stable manifold in the energy topology

We construct a center-stable manifold containing the ground state Q. All function spaces will be radial. Moreover, $B_\delta(Q)$ denotes a δ-ball in the energy space centered at Q. In contrast to Section 2, we work with the matrix formalism usually employed in asymptotic stability theory; the latter is of course closely related to the formalism of Section 2 but since we build upon [51], [20], [4], [5] it is convenient to adopt the complex-linear point of view. In what follows,

$$\mathcal{H}(\alpha,\gamma)=\begin{bmatrix}-\Delta+\alpha^2-2Q^2(\cdot,\alpha) & -e^{2i\gamma}Q^2(\cdot,\alpha)\\ e^{-2i\gamma}Q^2(\cdot,\alpha) & \Delta-\alpha^2+2Q^2(\cdot,\alpha)\end{bmatrix} \tag{7.1}$$

The following proposition constructs the center-stable manifold in a small neighborhood of Q. It should be compared to Definition 1.1; in fact, it provides much more detailed information than what is required by that definition. In Remark 7.1 we extend $\mathcal{M}$ so as to cover all of $\mathcal{S}$, and Corollary 7.2 characterizes the stable manifold, which lies in $\mathcal{M}$. A word about notation: henceforth, γ plays the role of a phase which has nothing to do with its previous usage, cf. (6.2).

For results on asymptotic stability analysis in the subcritical, and thus orbitally stable case, see Buslaev, Perelman [9], [10], Cuccagna [14], and Soffer, Weinstein [52], [53]. See also Pillet, Wayne [50].

Proposition 7.1. *There exists $\delta>0$ small and a smooth manifold $\mathcal{M}\subset H^1_{\rm rad}$ with the following properties: $\mathcal{S}\cap B_\delta(Q)\subset\mathcal{M}\subset B_\delta(Q)$, $\mathcal{M}$ divides $B_\delta(Q)$ into two connected components, and any initial data $u_0\in\mathcal{M}$ generates a solution of* (1.1) *for all $t\ge 0$ of the form*

$$u(x,t)=e^{i\theta(t)}Q(x,\alpha(t))+v(x,t)\quad\forall\, t\ge 0 \tag{7.2}$$

where $\theta(t)=-\int_0^t\alpha^2(s)\,ds+\gamma(t)$,

$$\begin{aligned}&\|\dot\gamma\|_{L^1\cap L^\infty(0,\infty)}+\|\dot\alpha\|_{L^1\cap L^\infty(0,\infty)}\lesssim\delta^2,\\ &\sup_{t\ge0}[|\alpha(t)-1|+|\gamma(t)|]\lesssim\delta\end{aligned} \tag{7.3}$$

The function v is small in the sense

$$\|v\|_{L^\infty_t((0,\infty);H^1(\mathbb{R}^3))}+\|v\|_{L^2_t((0,\infty);W^{1,6}(\mathbb{R}^3))}\lesssim\delta \tag{7.4}$$

and it scatters: $v(t)=e^{-it\Delta}v_\infty+o_{H^1}(1)$ as $t\to\infty$ for a unique $v_\infty\in H^1$.

$\mathcal{M}$ is unique in the following sense: there exists a constant C so that any $u_0\in B_\delta(Q)$ satisfies $u_0\in\mathcal{M}$ if and only if the solution $u(t)$ of (1.1) *with data u_0 has the property that $\operatorname{dist}(u(t),\mathcal{S}_1)\le C\delta$ for all $t\ge0$.*

Proof. Inserting (7.2) into (1.1) yields

$$i\partial_t\begin{pmatrix}v\\ \overline{v}\end{pmatrix}+\widetilde{\mathcal{H}}(t)\begin{pmatrix}v\\ \overline{v}\end{pmatrix}=\dot\gamma(t)\widetilde{\xi}(t)-i\dot\alpha(t)\widetilde{\eta}(t)+\widetilde{N}(t,v,\overline{v}) \tag{7.5}$$

where

$$\widetilde{\mathcal{H}}(t)=\begin{bmatrix}-\Delta-2Q^2(\cdot,\alpha(t)) & -e^{2i\theta(t)}Q^2(\cdot,\alpha(t))\\ e^{-2i\theta(t)}Q^2(\cdot,\alpha(t)) & \Delta+2Q^2(\cdot,\alpha(t))\end{bmatrix} \tag{7.6}$$

as well as

$$\widetilde{\xi}(t) = \begin{pmatrix} e^{i\theta(t)}Q(\cdot,\alpha(t)) \\ -e^{-i\theta(t)}Q(\cdot,\alpha(t)) \end{pmatrix}, \qquad \widetilde{\eta}(t) = \begin{pmatrix} e^{i\theta(t)}\partial_\alpha Q(\cdot,\alpha(t)) \\ e^{-i\theta(t)}\partial_\alpha Q(\cdot,\alpha(t)) \end{pmatrix} \tag{7.7}$$

and

$$\widetilde{N}(t,v,\overline{v}) = \begin{pmatrix} 2e^{i\theta(t)}Q(\cdot,\alpha(t))|v|^2 + e^{-i\theta(t)}Q(\cdot,\alpha(t))v^2 + |v|^2 v \\ -2e^{-i\theta(t)}Q(\cdot,\alpha(t))|v|^2 - e^{-i\theta(t)}Q(\cdot,\alpha(t))\overline{v}^2 - |v|^2\overline{v} \end{pmatrix} \tag{7.8}$$

Next, set

$$v(t) = e^{i\theta_0(t)}w(t), \qquad \theta_0(t) = -\int_0^t \alpha^2(s)\,ds. \tag{7.9}$$

Then (7.5) turns into

$$i\partial_t W + \mathcal{H}_\pi(t)W = \dot\gamma(t)\xi(t) - i\dot\alpha(t)\eta(t) + N_\pi(t,W) \tag{7.10}$$

where $W = \binom{w}{\overline{w}}$, and with $\pi = (\alpha,\gamma)$,

$$\mathcal{H}_\pi(t) = \begin{bmatrix} -\Delta + \alpha^2(t) - 2Q^2(\cdot,\alpha(t)) & -e^{2i\gamma(t)}Q^2(\cdot,\alpha(t)) \\ e^{-2i\gamma(t)}Q^2(\cdot,\alpha(t)) & \Delta - \alpha^2(t) + 2Q^2(\cdot,\alpha(t)) \end{bmatrix} \tag{7.11}$$

as well as

$$\xi(t) = \begin{pmatrix} e^{i\gamma(t)}Q(\cdot,\alpha(t)) \\ -e^{-i\gamma(t)}Q(\cdot,\alpha(t)) \end{pmatrix}, \qquad \eta(t) = \begin{pmatrix} e^{i\gamma(t)}\partial_\alpha Q(\cdot,\alpha(t)) \\ e^{-i\gamma(t)}\partial_\alpha Q(\cdot,\alpha(t)) \end{pmatrix} \tag{7.12}$$

and

$$N_\pi(t,W) = \begin{pmatrix} 2e^{i\gamma(t)}Q(\cdot,\alpha(t))|w|^2 + e^{-i\gamma(t)}Q(\cdot,\alpha(t))w^2 + |w|^2 w \\ -2e^{-i\gamma(t)}Q(\cdot,\alpha(t))|w|^2 - e^{i\gamma(t)}Q(\cdot,\alpha(t))\overline{w}^2 - |w|^2\overline{w} \end{pmatrix} \tag{7.13}$$

At this point we remark that all manipulations which we perform in this proof on (7.5) and (7.10) preserve the "admissible" subspace $\{\binom{f}{\bar f} \mid f : \mathbb{R}^3 \to \mathbb{C}\}$. This is necessary in order to return to the scalar formulation (7.2). In other words, the second row of these systems can be viewed as redundant, as it is always the complex conjugate of the first.

Let $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ be the third Pauli matrix, and set $\xi^*(t) = \sigma_3\xi(t)$, $\eta^*(t) = \sigma_3\eta(t)$. Impose the orthogonality conditions[3]

$$\langle W(t), \xi^*(t)\rangle = 0, \qquad \langle W(t), \eta^*(t)\rangle = 0 \qquad \forall t \ge 0 \tag{7.14}$$

Note that this imposes a condition on the data at $t = 0$. However, by the inverse function theorem there is a unique choice of $\alpha(0)$ and $\gamma(0)$ in a δ-neighborhood of $(1,0)$ so that (7.14) is satisfied; the needed nondegeneracy here is provided by $\langle Q|\partial_\alpha Q\rangle \neq 0$. Since $\mathcal{H}_\pi(t)^*\xi^*(t) = 0$ and $\mathcal{H}_\pi(t)^*\eta^*(t) = -2\alpha\xi^*(t)$, as well as $\langle \xi(t), \xi^*(t)\rangle = \langle \eta(t), \eta^*(t)\rangle = 0$, and $\langle \xi(t), \eta^*(t)\rangle = \langle \eta(t), \xi^*(t)\rangle = 2\langle Q|\partial_\alpha Q\rangle \neq 0$, one obtains from (7.10) that

$$\begin{aligned} \dot\gamma(t)\langle \xi(t), \eta^*(t)\rangle &= -i\langle W(t), \dot\eta^*(t)\rangle - \langle N_\pi(t,W), \eta^*(t)\rangle \\ -i\dot\alpha(t)\langle \eta(t), \xi^*(t)\rangle &= -i\langle W(t), \dot\xi^*(t)\rangle - \langle N_\pi(t,W), \xi^*(t)\rangle \end{aligned} \tag{7.15}$$

[3]Henceforth, we write $\langle\cdot,\cdot\rangle$ for the standard inner product in $L^2(\mathbb{R}^3;\mathbb{C}^2)$, whereas $\langle\cdot|\cdot\rangle$ is the inner product from Section 2.

The system (7.10), (7.14), (7.15) determines the evolution of $v(t), \alpha(t), \gamma(t)$ in (7.2). In fact, it suffices for (7.14) to hold at one point, say $t = 0$ since it then holds for all $t \geq 0$. More precisely, one needs to find a fixed point to this system consisting of a path $\pi(t) = (\alpha(t), \gamma(t))$ as well as a function $\binom{v}{\bar{v}}$, or equivalently, W satisfying the system as well as the bounds (7.3), (7.4).

We begin with the stability part of the underlying contraction argument, i.e., we turn (7.3) and (7.4) into bootstrap assumptions and then recover them from this system. Thus, suppose $\pi_0 = (\alpha_0, \gamma_0)$ and W_0 are given so that (7.3) and (7.4) hold and consider the following system of differential equations:

$$\begin{aligned} i\partial_t W + \mathcal{H}_{\pi_0}(t)W &= \dot{\gamma}(t)\xi_0(t) - i\dot{\alpha}(t)\eta_0(t) + N_{\pi_0}(t, W_0) \\ \dot{\gamma}(t)\langle \xi_0(t), \eta_0^*(t)\rangle &= -i\langle W(t), \dot{\eta}_0^*(t)\rangle - \langle N_{\pi_0}(t, W_0), \eta_0^*(t)\rangle \\ -i\dot{\alpha}(t)\langle \eta_0(t), \xi_0^*(t)\rangle &= -i\langle W(t), \dot{\xi}_0^*(t)\rangle - \langle N_{\pi_0}(t, W_0), \xi_0^*(t)\rangle \\ \langle W(0), \xi_0^*(0)\rangle &= 0, \quad \langle W(0), \eta_0^*(0)\rangle = 0 \end{aligned} \tag{7.16}$$

where $\mathcal{H}_{\pi_0}$, ξ_0, η_0 and $N_{\pi_0}(t, W_0)$ are defined as above but relative to the given functions π_0, W_0. The initial conditions are $\alpha(0) = \alpha_0(0)$, $\gamma(0) = \gamma_0(0)$; in addition to the final equation in (7.16), $W(0)$ needs to satisfy a further codimension-1 condition which will be specified below.

We begin with the $\dot{\alpha}, \dot{\gamma}$ part of (7.16). The W appearing on the right-hand side will be seen later to satisfy (7.4); for the moment, we will simply assume this bound. To be more specific, rewrite (7.3) and (7.4) in the form

$$\begin{aligned} \|\dot{\gamma}\|_{L^1\cap L^\infty} + \|\dot{\alpha}\|_{L^1\cap L^\infty} &\leq C_0\, \delta^2 \\ \|v\|_{L^\infty_t H^1(\mathbb{R}^3)} + \|v\|_{L^2_t W^{1,6}(\mathbb{R}^3)} &\leq C_1\, \delta \end{aligned} \tag{7.17}$$

and assume that $C_0 \gg C_1^2$. Inserting these bounds in the right-hand side of (7.15) yields

$$\|\dot{\gamma}\|_{L^1\cap L^\infty} + \|\dot{\alpha}\|_{L^1\cap L^\infty} \lesssim C_0 C_1\, \delta^3 + C_1^2\, \delta^2 \ll C_0\, \delta^2$$

provided δ is small. One can thus recover (7.17). The bound on v (or W) is more delicate. Since we are in the unstable regime, (7.10) is exponentially unstable. More precisely, write

$$\mathcal{H}_{\pi_0}(t) = \mathcal{H}_0 + a_0(t)\sigma_3 + D_0(t)$$

with the constant coefficient operator $\mathcal{H}_0 = \mathcal{H}(\alpha_0(0), \gamma_0(0))$, see (7.1), and $a_0(t) = \alpha_0^2(t) - \alpha_0^2(0)$, as well as $D_0(t)$ equaling

$$\begin{bmatrix} -2(Q^2(\cdot, \alpha_0(t)) - Q^2(\cdot, \alpha_0(0))) & -e^{2i\gamma_0(t)}Q^2(\cdot, \alpha_0(t)) + e^{2i\gamma_0(0)}Q^2(\cdot, \alpha_0(0)) \\ e^{-2i\gamma_0(t)}Q^2(\cdot, \alpha_0(t)) - e^{-2i\gamma_0(0)}Q^2(\cdot, \alpha_0(0)) & 2(Q^2(\cdot, \alpha_0(t)) - Q^2(\cdot, \alpha_0(0))) \end{bmatrix}$$

Note that $\|a_0(\cdot)\|_\infty \lesssim \delta^2$ and $\|\langle x\rangle^N D_0(\cdot)\|_\infty \lesssim \delta^2$ for any N provided the condition (7.3) holds.

Proposition B.1 in Section B details the spectral properties of $\mathcal{H}_0$ for the case $\alpha(0) = 1$ and $\gamma(0) = 0$. The more general case here follows by means of the rescaling $f \mapsto \alpha f(\alpha x)$, as well as a modulation by a constant unitary matrix. Following the notation of Proposition B.1 one writes

$$W(t) = \lambda_+(t)G_+ + \lambda_-(t)G_- + W_1(t) \tag{7.18}$$

where $\langle W_1(t), \sigma_3 G_\pm\rangle = 0$ for all $t \geq 0$. One needs to apply the aforementioned rescaling and modulation to $G_\pm$ and μ with the fixed parameters $\alpha_0(0), \gamma_0(0)$, which means that $\mu = \mu(\alpha_0(0))$, $G_\pm = G_\pm(\alpha_0(0), \gamma_0(0))$. We remark that $\lambda_\pm$ as defined in (7.18) are real-valued. Indeed, since $\|G_\pm\|_2 = 1$ and $G_+ = \binom{g_+}{\bar{g}_+} = \overline{G_-}$, the Riesz projections associated with the eigenvalues $\pm i\mu$ can be seen to be

$$P_\pm = \frac{\langle \cdot, \sigma_3 G_\mp\rangle}{\langle G_\pm, \sigma_3 G_\mp\rangle} G_\pm, \tag{7.19}$$

where $\langle G_\pm, \sigma_3 G_\mp\rangle \in i\mathbb{R} \setminus \{0\}$. Therefore,

$$\lambda_+ = \frac{\langle W, \sigma_3 G_-\rangle}{\langle G_\pm, \sigma_3 G_\mp\rangle} = \frac{2i\langle w|ig_-\rangle}{\langle G_\pm, \sigma_3 G_\mp\rangle} \in \mathbb{R} \tag{7.20}$$

and similarly for λ_-.
We now rewrite the W-equation in (7.16) in the form

$$\begin{aligned} &i\dot{\lambda}_+(t)G_+ + i\dot{\lambda}_-(t)G_- - i\mu\lambda_+(t)G_+ + i\mu\lambda_-(t)G_- \\ &\quad + i\partial_t W_1(t) + (\mathcal{H}_0 + a_0(t)\sigma_3)W_1 \\ &= -D_0(t)W - a_0(t)\sigma_3 G_+ - a_0(t)\sigma_3 G_- \\ &\quad + \dot{\gamma}(t)\xi_0(t) - i\dot{\alpha}(t)\eta_0(t) + N_{\pi_0}(t, W_0) =: F(t) \end{aligned} \tag{7.21}$$

Denote by $P_\pm$, P_0 the Riesz projections onto $G_\pm$, and the zero root space, respectively. Note that these operators are given by integration against exponentially decaying tensor functions. Moreover, we write

$$P_c = 1 - P_+ - P_- - P_0 = P_{\mathrm{real}} - P_0 \tag{7.22}$$

for the projection onto the continuous spectrum. Applying the projections $P_\pm$ to (7.21) yields the system of ODEs

$$\begin{aligned} (\dot{\lambda}_+(t) - \mu\lambda_+(t))G_+ &= ia_0(t)P_+(\sigma_3 W_1) - iP_+F(t) \\ (\dot{\lambda}_-(t) + \mu\lambda_-(t))G_- &= ia_0(t)P_-(\sigma_3 W_1) - iP_-F(t) \end{aligned} \tag{7.23}$$

For "generic" initial data $\lambda_+(0)$ the solution $\lambda_+(t)$ grows exponentially. However, there is a unique choice of initial condition that stabilizes λ_+ (i.e., ensures that it remains bounded) leading to the determination of the codimension one manifold. It is given by means of the following simple principle: suppose $\dot{x}(t) - \mu x(t) = f(t)$ with $f \in L^\infty((0,\infty))$. Then $x \in L^\infty((0,\infty))$ iff

$$0 = x(0) + \int_0^\infty e^{-\mu t} f(t)\, dt. \tag{7.24}$$

Thus,

$$0 = \lambda_+(0)G_+ + i\int_0^\infty e^{-t\mu}[a_0(t)P_+(\sigma_3 W_1)(t) - P_+F(t)]\, dt \tag{7.25}$$

is that unique choice. (7.25) has the following equivalent formulation

$$\lambda_+(t)G_+ = -i\int_t^\infty e^{-(s-t)\mu}[a_0(t)P_+(\sigma_3 W_1)(t) - P_+F(t)]\, ds \tag{7.26}$$

For $\lambda_-(t)$ we have the expression

$$\lambda_-(t)G_- = e^{-\mu t}\lambda_-(0)G_- + i\int_0^t e^{-(t-s)\mu}[a_0(t)P_-(\sigma_3 W_1)(t) - P_-F(t)]\, ds \tag{7.27}$$

Via (7.19) one checks that $\lambda_\pm$ as defined by these equations are real-valued. To determine the PDE for $W_1 = P_{\mathrm{real}}W_1 = P_{\mathrm{real}}W$, we write $W_1 = P_cW_1 + P_0W_1 = W_{\mathrm{disp}} + W_{\mathrm{root}}$. Then

$$i\partial_t W_{\mathrm{disp}}(t) + (\mathcal{H}_0 + a_0(t)\sigma_3)W_{\mathrm{disp}} = P_cF(t) - a(t)[\sigma_3, P_+ + P_- + P_0]W_1 \tag{7.28}$$

The sought after solution

$$(\alpha(t), \gamma(t), \lambda_+(t), \lambda_-(t), W_{\mathrm{root}}(t), W_{\mathrm{disp}}(t)) \tag{7.29}$$

is now determined from the second and third equations of (7.16), from (7.26), (7.27), and (7.28). The root part is controlled by the orthogonality conditions

$$\langle W, \xi_0^*\rangle(t) = \langle W, \eta_0^*\rangle(t) = 0 \quad \forall\, t \geq 0 \tag{7.30}$$

The main technical ingredient for the dispersive control of (7.28) is the Strichartz estimate of Lemma B.2, see Section B. The existence of the solution (7.29) is not entirely trivial since the determining equations contain these functions linearly on the right-hand side. However, they occur with small coefficients which allows one to iterate or contract; we skip those details. The solution obeys the estimates (7.3), (7.4). While (7.3) has already been established in this fashion, (7.4) is obtained as follows. Assuming again (7.17), one concludes from (7.26) and (7.27) that

$$\|\lambda_\pm\|_{L^\infty\cap L^2} \lesssim \delta + (C_0\, C_1\delta + C_0 + C_1^2)\delta^2 \ll C_1\,\delta \tag{7.31}$$

provided δ is sufficiently small. Via Lemma B.2 we conclude that $\|W_{\mathrm{disp}}\|_S \ll C_1\delta$ where S is the Strichartz space in (7.4). Finally, now that the path $(\alpha(t), \gamma(t))$ has been determined, as well as $\lambda_\pm(t)$, $W_{\mathrm{disp}}(t)$, the orthogonality conditions (7.14) determine W_{root} which also satisfies $\|W_{\mathrm{root}}\|_S \ll C_1\delta$. From these estimates, we conclude (7.4) via bootstrap as claimed.

The manifold is determined by (7.25) as a graph, *once a fixed point* $(\pi, W) = (\pi_0, W_0)$ is obtained. More precisely, for *fixed* $(\alpha(0), \gamma(0))$ we prescribe initial conditions $W^{(0)} \in H^1$, $\|W^{(0)}\|_{H^1} \lesssim \delta$ for (7.16) such that $P_0W^{(0)} = 0$ as well as $P_+W^{(0)} = 0$ where the projections are relative to $\mathcal{H}(\alpha(0), \gamma(0))$. Such data are linearly stable. The condition (7.25) takes nonlinear corrections into account and modifies the data in the form

$$W(0) = W^{(0)} + h(\pi_0, W_0, W^{(0)})G_+ \tag{7.32}$$

where $h(\pi_0, W_0, W^{(0)}) = \lambda_+(0)$ is real-valued and satisfies $|h(\pi_0, W_0, W^{(0)})| \lesssim \delta^2$. Since $\pi(0) = \pi_0(0)$ by construction, once we have found a fixed point, we can write $h(\pi_0, W_0, W^{(0)}) = h(\alpha_0(0), \gamma_0(0), W^{(0)})$ where the latter is smooth in $W^{(0)}$ in the sense of Fréchet derivatives. Moreover, the bound

$$|h(\alpha_0(0), \gamma_0(0), W^{(0)})| \lesssim \|W^{(0)}\|_{H^1}^2 \tag{7.33}$$

will hold. This shows that (7.32) describes a codimension-three manifold which is smoothly parametrized by $W^{(0)}$ and tangent to the subspace of linear stability. To regain the two missing codimensions, we vary $(\alpha_0(0), \gamma_0(0))$ in a δ-neighborhood

of $(1,0)$. In other words, we let the dilation and modulation symmetries act on the codimension-three manifold. Since these symmetries act transversely on the manifold (for the same reason that allowed us to enforce (7.14) at $t=0$ by modifying the data), we obtain a smooth codimension-one manifold which will be parametrized by

$$(\alpha_0(0),\gamma_0(0),W^{(0)})\in(1-\delta,1+\delta)\times(-\delta,\delta)\times B_\delta$$

where B_δ is a δ-ball in H^1. This is then the sought after $\mathcal{M}$.

Thus, one needs to find a fixed point for the system (7.16) via a contraction argument. The contraction argument is slightly delicate as it involves solving this system with two different but nearby *given* paths π_j^0, $j=0,1$ which therefore define different Hamiltonians via (7.11), and therefore also different orthogonality conditions (7.14). Note that phases of the form $t\alpha^2$ and $t\tilde{\alpha}^2$ diverge linearly if $\alpha^2\neq\tilde{\alpha}^2$. This make it necessary to employ a weaker norm than the one used in the previous stability argument, see (7.3), (7.4).

In order to carry out the comparison between two solutions, we work on the level of (7.5) rather than with the aforementioned W-system. Thus, consider two paths $\pi_j^0(t)=(\alpha_j^0(t),\gamma_j^0(t))$ satisfying (7.3) and with $\pi_0^{(0)}(0)=\pi_1^{(0)}(0)$, and the associated equations with $Z=\binom{v}{\overline{v}}$

$$i\partial_t Z+\widetilde{\mathcal{H}}_j(t)Z=\dot{\gamma}_j(t)\widetilde{\xi}_j(t)-i\dot{\alpha}_j(t)\widetilde{\eta}_j(t)+\widetilde{N}_j(t,v_j^0,\overline{v}_j^0) \tag{7.34}$$

for $j=0,1$, see (7.5). Here $\widetilde{\mathcal{H}}_j$, $\widetilde{\xi}_j$, $\widetilde{\eta}_j$ and $\widetilde{N}_j(t,v,\overline{v})$ are defined as in (7.6), (7.7), (7.8) but relative to the paths $\pi_j^0(t)$. Moreover, the function v_j^0 are given and satisfy (7.4), and we impose the orthogonality conditions, see (7.14),

$$\langle Z(t),\sigma_3\widetilde{\xi}_j(t)\rangle=\langle Z(t),\sigma_3\widetilde{\eta}_j(t)\rangle=0\qquad\forall\, t\geq 0 \tag{7.35}$$

The initial conditions for the paths are $\pi_0(0)=\pi_1(0)=\pi_0^{(0)}(0)$, whereas for Z_0,Z_1 one invokes (7.32) as follows: fix $Z_0^{(0)}\in B_\delta(0)$ so that $P_0Z_0^{(0)}=P_+Z_0^{(0)}=0$ and set

$$\begin{aligned}Z_0(0)&=Z_0^{(0)}+h(\pi_0^0,W_0^0,W_0^{(0)})G_+\\ Z_1(0)&=Z_0^{(0)}+h(\pi_1^0,W_1^0,W_0^{(0)})G_+\end{aligned} \tag{7.36}$$

This choice guarantees that (7.35) holds at $t=0$. By the preceding stability analysis, (7.34) and (7.35) then define unique solutions (π_j,Z_j) satisfying (7.3) and (7.4). Differentiating (7.35) in combination with (7.34) yields the modulation equations (7.15). Thus, we rewrite (7.34) in the form

$$i\partial_t Z_j+\widetilde{\mathcal{H}}_j(t)Z_j=-iL_j(t)Z_j+N_j(t,v_j^0,\overline{v}_j^0) \tag{7.37}$$

where N_j incorporates both $\widetilde{N}_j$ and the nonlinear term in (7.15). The linear term $L_j(t)Z_j$ is of finite rank and corank, and satisfies the estimates

$$\|L_j(t)Z_j\|_{W^{k,p}}\lesssim\|Z_j\|_{H^1}|\dot{\pi}_j^0(t)| \tag{7.38}$$

for any $1\leq p\leq\infty$ and $k\geq 0$. Combining this pointwise in time bound with (7.3) yields the full estimates on $L_j(t)Z_j$. By construction, any solution of (7.37) which satisfies (7.35) at one point, say $t=0$, satisfies (7.14) for all $t\geq 0$.

The difference $R := Z_1 - Z_0$ satisfies

$$\begin{aligned} i\partial_t R + \widetilde{\mathcal{H}}_0(t)R &= -iL_1(t)R - N_0(t, v_0^0, \overline{v}_0^0) + N_1(t, v_1^0, \overline{v}_1^0) \\ &\quad + (\widetilde{\mathcal{H}}_0(t) - \widetilde{\mathcal{H}}_1(t))Z_1 - i(L_1(t) - L_0(t))Z_0 =: \widetilde{F} \end{aligned} \tag{7.39}$$

whereas the difference of the paths $\pi = \pi_1 - \pi_0$ is governed by taking differences of the third and fourth equations, respectively, in (7.16) for $j = 1, 0$. We estimate (R, π) in the norm, with $\rho > 0$ small, fixed, and to be determined,

$$\|(R,\pi)\|_Y := \|e^{-t\rho}R\|_{L_t^\infty((0,\infty);L^2)} + \|e^{-t\rho}\dot{\pi}\|_{L^1((0,\infty))} \tag{7.40}$$

To render this a norm, one fixes $\pi(0) = 0$, say. Note that some measure of growth has to be built into $\|\cdot\|_Y$, since $\widetilde{\mathcal{H}}_0(t) - \widetilde{\mathcal{H}}_1(t)$ and $L_1(t) - L_0(t)$ grow linearly in t. Next, we perform the same modulation as above, i.e.,

$$W(t) := \begin{bmatrix} e^{-i\theta_0^0(t)} & 0 \\ 0 & e^{i\theta_0^0(t)} \end{bmatrix} R, \qquad \theta_0^0(t) = -\int_0^t (\alpha_0^0(s))^2\, ds$$

Denoting the matrix here by $M_0(t)$, W satisfies the equation

$$i\partial_t W + (\widetilde{\mathcal{H}}_0(t) + (\alpha_0^0(0))^2\sigma_3)W = M_0\widetilde{F} \tag{7.41}$$

To obtain estimates on (7.41), we write

$$\widetilde{\mathcal{H}}_0(t) + (\alpha_0^0(0))^2\sigma_3 = \mathcal{H}_0^0 + a(t)\sigma_3 + D(t)$$

with the constant coefficient operator $\mathcal{H}_0^0 = \mathcal{H}(\alpha_0^0(0), \gamma_0^0(0))$, see (7.1), and $a(t) = (\alpha_0^0(t))^2 - (\alpha_0^0(0))^2$, as well as $D(t)$ equaling

$$\begin{bmatrix} -2(Q^2(\cdot,\alpha_0^0(t)) - Q^2(\cdot,\alpha_0^0(0))) & -e^{2i\gamma_0^0(t)}Q^2(\cdot,\alpha_0^0(t)) + e^{2i\gamma_0^0(0)}Q^2(\cdot,\alpha_0^0(0)) \\ e^{-2i\gamma_0^0(t)}Q^2(\cdot,\alpha_0^0(t)) - e^{-2i\gamma_0^0(0)}Q^2(\cdot,\alpha_0^0(0)) & 2(Q^2(\cdot,\alpha_0^0(t)) - Q^2(\cdot,\alpha_0^0(0))) \end{bmatrix}$$

One has $\|a(\cdot)\|_\infty \lesssim \delta^2$ and $\|\langle x\rangle^N D(\cdot)\|_\infty \lesssim \delta^2$ for any N as before. At this point the analysis is similar to the one starting with (7.18). Indeed, writing once again

$$W = \lambda_+ G_+ + \lambda_- G_- + W_{\mathrm{root}} + W_{\mathrm{disp}}$$

where the decomposition is carried out relative to $\mathcal{H}_0^0$, one inserts this into (7.41) and proceeds as before. The two main differences from the previous stability analysis are as follows: (i) the stability condition (7.25) holds automatically here, since we know apriori that λ_+ remains bounded; indeed, we chose Z_1, Z_0 to each satisfy (7.25) whence (7.4) holds for each of these functions. (ii) the orthogonality condition (7.14) does not hold exactly in this form, since it is obtained by taking the difference of the orthogonality conditions satisfied by Z_1 and Z_0. But this is minor, since the error one generates in this fashion is contractive.

Applying the dispersive bound of Lemma B.2 (here we need only the L_x^2 part) to W_{disp} yields via a term-wise estimation of the right-hand side of (7.39),

$$\begin{aligned} \|R(t)\|_2 &\lesssim \delta e^{t\rho}\|(R_0^0 - R_1^0, \pi_0^0 - \pi_1^0)\|_Y + \delta\int_0^t \|R(s)\|_2\, ds \\ &\quad + \delta\int_t^\infty e^{-\mu(s-t)}\|R(s)\|_2\, ds \end{aligned} \tag{7.42}$$

where $R_j^0 = \binom{v_j^0}{\bar{v}_j^0}$ for $j = 0, 1$. Recall that the initial conditions for R are determined by (7.36). The final integral in (7.42) is a result of the λ_+ equation (7.26). Assuming $\mu(\alpha_0^0(0)) > \rho \gg \delta$, Gronwall's inequality implies

$$\sup_{t\geq 0} e^{-t\rho}\|R(t)\|_2 \lesssim \delta\rho^{-1}\|(v_0^0 - v_1^0, \pi_0^0 - \pi_1^0)\|_Y \tag{7.43}$$

as desired. Next, one estimates the π equation with initial condition $\pi(0) = 0$. The conclusion is a bound of the form

$$\|(R,\pi)\|_Y + |h(\pi_0^0, W_0^0, W_0^{(0)}) - h(\pi_1^0, W_1^0, W_0^{(0)})| \ll \|(v_0^0 - v_1^0, \pi_0^0 - \pi_1^0)\|_Y \tag{7.44}$$

which proves the desired contractivity. See [5] for more details on these estimates. Hence, one has a fixed point of (7.16) as well as a well-defined function $h(\pi_0^0(0), W_0^{(0)})$. This concludes the proof of the existence part.

Next, we turn to scattering. In contrast to the previous analysis, we do not linearize around $\mathcal{H}(\alpha(0),\gamma(0))$, but rather around $\mathcal{H}(\alpha(\infty),\gamma(\infty))$. Thus, consider the system (7.15), (7.14), (7.26), (7.27), (7.28) with $a(t) = \alpha^2(t) - \alpha^2(\infty)$, F defined by (7.21), and $D(t)$ equaling

$$\begin{bmatrix} -2(Q^2(\cdot,\alpha(t)) - Q^2(\cdot,\alpha(\infty))) & -e^{2i\gamma(t)}Q^2(\cdot,\alpha(t)) + e^{2i\gamma(\infty)}Q^2(\cdot,\alpha(\infty)) \\ e^{-2i\gamma(t)}Q^2(\cdot,\alpha(t)) - e^{-2i\gamma(\infty)}Q^2(\cdot,\alpha(\infty)) & 2(Q^2(\cdot,\alpha(t)) - Q^2(\cdot,\alpha(\infty))) \end{bmatrix} \tag{7.45}$$

Thus, $a(t), D(t) \to 0$ as $t \to \infty$. This ensures the vanishing at $t = \infty$ of the first three terms of $F(t)$ in (7.21). The fourth and fifth terms of F vanish in the $L^1(T,\infty)$-sense as $T \to \infty$ by (7.3), whereas the nonlinear term $N(t,W)$ vanishes in the sense of Strichartz estimates. Therefore, (7.26), (7.27) imply that $\lambda_\pm(t) \to 0$ as $t \to \infty$. Hence, in view of the scattering statement in Lemma B.2 one has the representation in H^1

$$\begin{aligned} W_{\mathrm{disp}}(t) &= e^{i\sigma_3\int_0^t(-\Delta+\alpha^2(\infty)+a(s))\,ds}W_\infty + o(1) \\ &= e^{i\sigma_3\int_0^t(-\Delta+\alpha^2(s))\,ds}W_\infty + o(1) \end{aligned} \tag{7.46}$$

as $t \to \infty$. The modulation in (7.9) removes the $\alpha^2(s)$ in the exponent once we return to the v representation. Finally, by the orthogonality conditions $W_{\mathrm{root}}(t) \to 0$. In summary, we have obtained the desired scattering statement for v in (7.2).

Finally, to obtain the uniqueness statement let $u(t)$ be a solution with $u(0) \in B_\delta(Q)$ and with the property that $\mathrm{dist}(u(t),\mathcal{S}_1) \lesssim \delta$ for all $t \geq 0$. We claim that there exists a C^1-curve $(\alpha(t),\theta(t)) \in (1-O(\delta), 1+O(\delta)) \times \mathbb{R}$ which achieves

$$\begin{aligned} \langle u(t) - e^{i\theta(t)}Q(\cdot,\alpha(t)) | e^{i\theta(t)}Q(\cdot,\alpha(t))\rangle &= 0 \\ \langle u(t) - e^{i\theta(t)}Q(\cdot,\alpha(t)) | ie^{i\theta(t)}\partial_\alpha Q(\cdot,\alpha(t))\rangle &= 0 \end{aligned} \tag{7.47}$$

for all $t \geq 0$, as well as

$$\sup_{t\geq 0}\|u(t) - e^{i\theta(t)}Q(\cdot,\alpha(t))\|_{H^1} \lesssim \delta \tag{7.48}$$

In fact, by definition there is a C^1–path $\tilde{\theta}(t) \in \mathbb{R}$ so that

$$\sup_{t\geq 0}\|u(t) - e^{i\tilde{\theta}(t)}Q(\cdot,1)\|_{H^1} \lesssim \delta$$

This shows that one can fulfill (7.47) up to $O(\delta)$. Next, one uses that $|\langle Q|\partial_\alpha Q\rangle| \simeq 1$ for all $\alpha \simeq 1$ and the inverse function theorem to show that $(\tilde{\alpha} \equiv 1, \tilde{\theta})$ can be modified by an amount $O(\delta)$ so as to exactly satisfy (7.47) without violating (7.48). Furthermore, by chaining one concludes that this procedure yields a well-defined path (α, θ) which is C^1, as claimed. Next, define

$$\theta_0(t) = -\int_0^t \alpha^2(s)\,ds + \theta(0)$$

and set $\gamma = \theta - \theta_0$. Now write

$$u(t) = e^{i\theta(t)}Q(\cdot,\alpha(t)) + v(t) = e^{i\theta(t)}Q(\cdot,\alpha(t)) + e^{i\theta_0(t)}w(t) \tag{7.49}$$

This then allows one to rewrite (7.47) in the form

$$\langle e^{i\gamma(t)}Q(\cdot,\alpha(t))|w(t)\rangle = 0, \qquad \langle ie^{i\gamma(t)}\partial_\alpha Q(\cdot,\alpha(t))|w(t)\rangle = 0 \tag{7.50}$$

As before, consider $W = \binom{w}{\overline{w}}$, and perform the decomposition (7.18). Inserting (7.49) into (1.1) yields, cf. (7.5),

$$i\partial_t\begin{pmatrix} v \\ \overline{v}\end{pmatrix} + \widetilde{\mathcal{H}}(t)\begin{pmatrix} v \\ \overline{v}\end{pmatrix} = (\dot{\theta}(t) + \alpha^2(t))\widetilde{\xi}(t) - i\dot{\alpha}(t)\widetilde{\eta}(t) + \widetilde{N}(t,v,\overline{v}) \tag{7.51}$$

where $\widetilde{\mathcal{H}}, \widetilde{\xi}, \widetilde{\eta}$, and $\widetilde{N}$ are as in (7.6), (7.7), (7.8). Furthermore, with $W = \binom{w}{\overline{w}}$,

$$i\partial_t W + \mathcal{H}(t)W = \dot{\gamma}(t)\xi(t) - i\dot{\alpha}(t)\eta(t) + N(t,W) \tag{7.52}$$

see (7.10), (7.12), (7.13). The orthogonality conditions (7.50) are of the form

$$\langle W(t), \xi^*(t)\rangle = 0, \qquad \langle W(t), \eta^*(t)\rangle = 0 \tag{7.53}$$

which is identical with (7.14). This places us in the exact same position that we started from in the existence proof. Thus, the decomposition (7.49) is such that (7.3) and (7.4) hold. The only difference here is that we know apriori that $\lambda_+(t)$ is bounded. However, (7.24) guarantees that therefore (7.25) holds which forces the solution to lie on $\mathcal{M}$ as desired. $\square$

Remark 7.1. Denote the manifold constructed in Proposition 7.1 by $\mathcal{M}_{1,0}$. The same construction can be applied to $e^{i\gamma}Q(x,\alpha)$ instead of Q for any $\gamma \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\alpha > 0$, yielding a codimension one manifold in the phase space $\mathcal{H}$ which we denote by $\mathcal{M}_{\alpha,\gamma}$. By the uniqueness part of Proposition 7.1 one concludes that

$$\mathcal{M}_{\mathcal{S}} := \bigcup_{\alpha>0,\gamma\in\mathbb{T}} \mathcal{M}_{\alpha,\gamma} \tag{7.54}$$

is again a smooth manifold, which contains all of $\mathcal{S}$. By the proof of Proposition 7.1 it is smoothly parametrized by $(\alpha(0), \gamma(0), W(0))$ where $P_0(\alpha(0),\gamma(0))W(0) = 0$ and $P_+(\alpha(0),\gamma(0))W(0) = 0$ and $W(0)$ needs to be small enough.

It has the property that any $u_0 \in \mathcal{M}_{\mathcal{S}}$ leads to a solution of (1.1) defined on $t \geq 0$ which scatters to $\mathcal{S}$ as $t \to \infty$ in the sense of Definition 1.1. We emphasize that this is *not* the manifold $(5)\cup(7)\cup(9)$ appearing in Theorem 1.2. Rather, that manifold is the maximal backward evolution of $\mathcal{M}_{\mathcal{S}}$ under the NLS flow. Note that $\mathcal{M}_{\mathcal{S}}$, thus extended by the nonlinear flow, is again a manifold.

The following characterization of the stable manifolds will be needed in the proof of Theorem 1.3. It precisely captures the case where the radiation part (i.e., the difference between $u(t)$ and the soliton in (1.6)) has vanishing scattering data and is therefore uniquely captured by $\lambda_-(0)$.

Corollary 7.2. *Let $\mathcal{M}_{\mathcal{S}}$ be as in (7.54). Suppose $u_0 \in \mathcal{M}_{\mathcal{S}}$ with $M(u_0) = M(Q)$ forward scatters to $\mathcal{S}$ in the sense of Definition 1.1 so that (1.6) holds with $u_\infty = 0$. Then the solution $u(t)$ of (1.1) with data u_0 approaches a soliton trajectory in $\mathcal{S}_1$ exponentially fast. Moreover, the solution is uniquely characterized by $\gamma_\infty \in S^1$ and a real number λ_0 with $|\lambda_0| \lesssim \delta$. The case where u is an exact soliton is characterized by $\lambda_0 = 0$.*

Proof. This follows from the construction carried out in the proof of Proposition 7.1, but with $\mathcal{H}_0 = \mathcal{H}(\alpha(\infty), \gamma(\infty))$ as the driving linear operator; see that part of the proof dealing with scattering. By (1.7), $\alpha(\infty) = 1$. In fact, consider the representation
$$W = \lambda_+ G_+ + \lambda_- G_- + W_{\mathrm{root}} + W_{\mathrm{disp}}$$
relative to this choice of $\mathcal{H}_0$, and solve the system (7.15), (7.14), (7.26), (7.27), (7.28) with $a(t) = \alpha^2(t) - \alpha^2(\infty)$, F defined by (7.21), and $D(t)$ given by (7.45). For (7.15) one assigns the terminal conditions $\alpha(\infty) = \alpha_\infty = 1$, $\gamma(\infty) = \gamma_\infty$, for (7.27) we impose the initial conditions $\lambda_-(0) = \lambda_0$, and (7.28) is solved with scattering data $W_\infty = 0$, cf. (7.46). Note that λ_+ does not require any further data, see (7.26). Similarly, W_{root} is determined by (7.14). The point is that we can solve the aforementioned system for $(\alpha(t), \gamma(t))$, and $W(t)$ satisfying (7.3), (7.4) by contracting in the *strong* norm
$$\|(W, \pi)\|_Y := \|e^{t\rho} W\|_{L^\infty_t((0,\infty);L^2)} + \|e^{t\rho} \dot{\pi}\|_{L^1((0,\infty))} \tag{7.55}$$
for suitably chosen and small $\rho > 0$. Note the contrast to (7.40). In the case of (7.40) the exponentially decaying weights forced us to start from $t = 0$ when carrying out the contraction argument. In the case of (7.55), however, we can solve for W_{disp} from $t = \infty$ due to the exponentially growing weights. It is essential, though, that for λ_- we can still start at $t = 0$; this is due to the fact that equation (7.27) contains exponentially decreasing functions (one therefore needs $\rho < \mu$ but nothing else). In summary, $\pi(t) - \pi(\infty)$ decreases exponentially, as do $a(t)$, $D(t)$, $\lambda_\pm$, W_{disp}, W_{root}. This proves the exponential approach to $\mathcal{S}_1$. Since $\alpha^2(t) - \alpha^2(\infty) \to 0$ and $\gamma(t) - \gamma(\infty) \to 0$ at an exponential rate, $u(t)$ in fact converges to a soliton trajectory in $\mathcal{S}_1$ exponentially fast. The case of an exact soliton is given by $W = 0$, which the contraction argument characterizes as $\lambda_-(0) = \lambda_0 = 0$. $\square$

8. Proof of Theorems 1.2, 1.3

Proof of Theorem 1.2. We may rescale any solution to mass one. If u is trapped by $\mathcal{S}_1$, then provided $\varepsilon \ll \delta$, where the latter is from the previous section, one concludes from the uniqueness part of Proposition 7.1 that $u \in \mathcal{M}_{\mathcal{S}}$ for large times (see Remark 7.1 for the definition of $\mathcal{M}_{\mathcal{S}}$). Conversely, every solution starting on $\mathcal{M}_{\mathcal{S}}$ is trapped. Therefore, the set $(5) \cup (7) \cup (9)$ is the maximal backward evolution of $\mathcal{M}_{\mathcal{S}}$, see Remark 7.1, whereas $(6) \cup (8) \cup (9)$ is the maximal forward evolution of $\overline{\mathcal{M}_{\mathcal{S}}}$ (complex conjugate). If we reverse time and conjugate, then the stable

and unstable modes $\lambda_\pm$ are interchanged. This means that the intersection of the center-stable manifold as $t \to \infty$ with the corresponding one for $t \to -\infty$ intersect transversely in a smooth manifold of codimension two, i.e., the center manifold. $\square$

Proof of Theorem 1.3. Let u be a solution with $M(u) = M(Q)$. By assumption, $E(u) \le E(Q)$. If $E(u) < E(Q)$, then [31] show that either u scatters at $\pm\infty$, or blows up in finite time in both directions. Therefore, assume that $E(u) = E(Q)$. If u blows up in finite negative time, then $K(u(t)) < 0$ for some $t < 0$. If u were to scatter at $t = +\infty$, then $K(u(t)) > 0$ for some $t > 0$. But then $K(u(t_0)) = 0$ for some t_0, which implies that $u(t) = e^{i(-t+\theta_0)}Q$ which is a contradiction. Thus, the sets (3) and (4) are empty. Now suppose u is trapped by $\mathcal{S}$ as $t \to \infty$. Then for large times u needs to lie on $\mathcal{M}_{1,\gamma}$ for some γ, see (1.7); in particular, $\alpha_\infty = 1$. Since $E(u) = E(Q)$, it follows from Corollary 7.2 that u is uniquely described by $(\gamma_\infty, \lambda_0)$. Fixing the symmetry parameter γ_∞, we see that the solution is described by the single real-valued parameter λ_0. If $\lambda_0 = 0$, then necessarily $u(t) = e^{i(-t+\gamma_\infty)}Q$. Therefore, $\lambda_0 \ne 0$ and the sign of this parameter uniquely determines the sign of $K(u(t))$ upon ejection as in Lemma 3.3, deciding the fate of $u(t)$ for negative times. This shows that one obtains two one-dimensional manifolds which approach soliton trajectories from $\mathcal{S}_1$ in the H^1-norm as $t \to \infty$ exponentially fast, but which either blow up in finite negative time, or scatter to zero as $t \to -\infty$. Since time-translation leaves these manifolds invariant, it follows that they have the form described in Theorem 1.3. $\square$

Appendix A. Some tools from scattering theory

The NLS equation (1.1) is subcritical relative to $H^1(\mathbb{R}^3)$ and critical relative to $\dot H^{\frac12}(\mathbb{R}^3)$. There is a classical local well-posedness theory for (1.1) for data of these regularity classes, see [11] as well as [31]. We work on the level of H^1. As usual, we say that (q, r) is Strichartz admissible in $\mathbb{R}^3$ if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$$

and we set, using [34],

$$\|u\|_S = \sup_{\substack{(q,r)\ \text{admissible}\\ 2\le r\le 6,\ 2\le q\le\infty}} \|u\|_{L^q_t L^r_x}$$

The following small data scattering lemma is standard, and we leave the proof the reader.

Lemma A.1. *Let u be a solution of* (1.1) *in $\mathbb{R}^3$ on a time-interval I. If there exists $t_0 \in I$ with $\|\nabla u(t_0)\|_2 \le \mu$ where μ is a constant satisfying $\mu M(u)^{1/2} \ll 1$, then u extends to a global solution satisfying the global Strichartz bound*

$$\|\nabla u\|_{L^\infty_t L^2_x} + \|\nabla u\|_{L^2_t L^6_x} \lesssim \mu \tag{A.1}$$

as well as

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^2_t L^6_x} \lesssim M(u)^{1/2}$$

In particular, u scatters: there exists $u_0 \in H^1$ with $\|u(t) - e^{-it\Delta}u_0\|_{H^1} \to 0$ as $|t| \to \infty$.

For the cubic equation, one has the following version of the linear Bahouri-Gérard profile decomposition, see [1], [37], as well as Lemma 5.2 of [31] and Proposition 6.1 of [47]. All function spaces are radial.

Proposition A.2. *Let $\{u_n\}_{n=1}^\infty$ be a bounded sequence in H^1. Then there exist a sequence $\{v^j\}_{j=0}^\infty$ bounded in H^1, and sequences of times $t_n^j \in \mathbb{R}$ such that for any $k \geq 1$ one has the following property, after replacing $\{u_n\}_{n=1}^\infty$ by a subsequence: let γ_n^k defined by*

$$e^{-it\Delta}u_n = \sum_{0\leq j<k} e^{-i(t+t_n^j)\Delta}v^j + \gamma_n^k(t) \tag{A.2}$$

we have for any $0 \leq j < k$, $\gamma_n^k(-t_n^j) \rightharpoonup 0$ weakly in H^1 as $n \to \infty$, as well as

$$\lim_{k\to\infty}\limsup_{n\to\infty} \|\gamma_n^k\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^3)} = 0, \quad \lim_{n\to\infty}|t_n^j - t_n^k| = \infty \tag{A.3}$$

where (p, q) is any pair which can be obtained by interpolation[4] of (some nonzero amount of) $L_t^\infty L_x^3$ with $L_t^2 W_x^{1,6} \cap L_t^\infty H_x^1$. In particular, $(p, q) = (\infty, 3)$ as well as $(\infty, 4), (4, 4)$ are such choices. Moreover, one has the following partition of the H^1-norm:

$$\limsup_{n\to\infty}\Big|\|u_n\|_{H^1}^2 - \sum_{j<k}\|v^j\|_{H^1}^2 - \|\gamma_n^k\|_{H^1}^2\Big| = 0, \tag{A.4}$$

and the same holds for L^2.

Proof. One has $u_n \rightharpoonup v^0$ in H^1 (henceforth, we pass to subsequences without further mention). By the compact radial imbedding $H^1 \hookrightarrow L^p$ for $2 < p < 6$ one then has strong convergence in L^4. Set $t_n^0 = 0$. Passing to $u_n - v^0$, we may assume that $u_n \rightharpoonup 0$. Clearly, $\gamma_n^1(t) = e^{-it\Delta}u_n$ satisfies $\gamma_n^1(0) \rightharpoonup 0$ as claimed. If

$$\liminf_{n\to\infty}\|e^{-it\Delta}u_n\|_{L_t^\infty L_x^3} = 0, \tag{A.5}$$

then the process terminates. Otherwise, pick $t_{1,n}$ so that L_t^∞ in (A.5) is attained at those times. Then $e^{-it_{1,n}\Delta}u_n \rightharpoonup v^1 \neq 0$ by the aforementioned compact imbedding. Since $u_n \rightharpoonup 0$, we must have $|t_{1,n}| \to \infty$ as $n \to \infty$. The process now repeats inductively in a standard way, see for example [31], [47]. □

Next, one has the following perturbation lemma, cf. Lemma 6.2 in [47], and Proposition 2.3 in [31].

Lemma A.3. *There are continuous functions $\nu_0, C_0 : (0,\infty)^2 \to (0,\infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an interval, $u, w \in C(I; H^1)$ satisfying for some $A, B > 0$ and $t_0 \in I$*

$$\|u\|_{L_t^\infty(I;H^1)} + \|w\|_{L_t^\infty(I;H^1)} \leq A, \quad \|w\|_{L_t^4(I;L_x^4)} \leq B, \tag{A.6}$$

$$\|eq(u)\|_{L_t^{\frac{8}{5}}(I;L_x^{\frac{4}{3}})} + \|eq(w)\|_{L_t^{\frac{8}{5}}(I;L_x^{\frac{4}{3}})} + \|\gamma_0\|_{L_t^{\frac{8}{3}}(I;L_x^4)} \leq \nu \leq \nu_0(A, B), \tag{A.7}$$

where $eq(u) := i\partial_t u - \Delta u - |u|^2 u$ and similarly for w, and $\gamma_0 := e^{-i(t-t_0)\Delta}(u-w)(t_0)$. Then we have

$$\|u - w - \gamma_0\|_{L_t^\infty L_x^2(I)} \leq C_0(A, B)\nu, \quad \|u - w\|_{L_{t,x}^4(I)} \leq C_0(A, B)\nu^{1/3}. \tag{A.8}$$

[4]One could use here $L_t^\infty L_x^p$ for $4 > p > 2$.

Proof. We fix a L^2-admissible Strichartz space $Z := L_t^{\frac{8}{3}}(I; L_x^4)$ and

$$\gamma := u - w, \quad e := (i\partial_t - \Delta)(u - w) - |u|^2 u + |w|^2 w = eq(u) - eq(w).$$

There exists a partition of $I \cap [t_0, \infty)$ such that

$$\begin{gathered} t_0 < t_1 < \cdots < t_n \le \infty, \quad I_j = (t_j, t_{j+1}), \quad I \cap (t_0, \infty) = (t_0, t_n), \\ \max_{0 \le j < n} \|w\|_{L_t^4(I_j; L_x^4)} \le \delta, \quad n \le C(B, \delta). \end{gathered}$$

We omit the estimate on $I \cap (-\infty, t_0)$ since it is the same by symmetry. Let $\gamma_j(t) := e^{-i(t-t_j)\Delta}\gamma(t_j)$. Then the Strichartz estimate applied to the equations of γ and γ_{j+1} implies

$$\begin{aligned} \|\gamma - \gamma_j\|_{Z(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{Z(\mathbb{R})} &\lesssim \||w + \gamma|^2(w + \gamma) - |w|^2 w + e\|_{L_t^{\frac{8}{5}}(I_j; L_x^{\frac{4}{3}})} \\ &\lesssim A\delta\|\gamma\|_{Z(I_j)} + A^{\frac{1}{3}}\|\gamma\|_{Z(I_j)}^{\frac{5}{3}} + \nu, \end{aligned} \tag{A.9}$$

where in the second step the Hölder inequality was used in t and in x, together with the Sobolev $H_x^1 \subset L_x^4$. Hence by induction on j and continuity in t, one obtains provided $A\delta \ll 1$,

$$\|\gamma\|_{Z(I_j)} + \|\gamma_{j+1}\|_{Z(I)} \le (2C)^j \nu \le (2C)^n \nu \ll \delta, \tag{A.10}$$

provided that $\nu_0(A, B)$ is chosen small enough. Repeating the estimate (A.9) once more, we can bound $\|\gamma - \gamma_0\|_{L_t^\infty L_x^2}$ as well. The bound in $L_{t,x}^4$ is obtained by the interpolation

$$\|u\|_{L_{t,x}^4} \lesssim \|u\|_{L_t^4 W^{1/3,3}} \lesssim \|u\|_{L_t^\infty H_x^1}^{1/3} \|u\|_Z \tag{A.11}$$

and we are done. $\square$

Appendix B. Spectral properties and linear dispersive estimates

We begin with a result on the spectral properties of $\mathcal{H}$, see (7.1) with $\alpha(0) = 1$, $\gamma(0) = 0$. As usual, $Q = Q(\cdot, 1)$ for simplicity. We view all operators in this section as complex linear ones. Then

$$\mathcal{H} = \begin{bmatrix} -\Delta + 1 - 2Q^2(\cdot, 1) & -Q^2(\cdot, 1) \\ Q^2(\cdot, 1) & \Delta - 1 + 2Q^2(\cdot, 1) \end{bmatrix}$$

is conjugate to

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \mathcal{H} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = i \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \tag{B.1}$$

where

$$L_+ = -\Delta + 1 - 3Q(\cdot, 1)^2, \qquad L_- = -\Delta + 1 - Q(\cdot, 1)^2$$

The equality (B.1) is to be considered as one between complex linear operators. However, it is also natural to view the left-hand side as acting on vectors $\binom{u_1}{u_2}$ with u_1, u_2 real-valued. In that case the right-hand side needs to be rewritten as

$$\begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} = \mathcal{L}$$

This is exactly the point of view taken in Section 2, where $\mathcal{L}$ is considered as a real-linear operator. The spectral properties of L_+, L_- and especially $\mathcal{H}$ are

quite delicate. The following result summarizes what can be obtained by a rigorous analysis, see [51], [20], [32], supported by numerics, such as [16] and [41]. In the work of Marzuola, Simpson [41] numerics is used to assist index computations of certain quadratic forms in the spirit of the virial argument of Fibich, Merle, Raphael [21].

For simplicity, we restrict ourselves to the Hilbert space[5] $L^2_{\mathrm{rad}}(\mathbb{R}^3)$ in Proposition B.1.

Proposition B.1. *The essential spectrum of $\mathcal{H}$ is $(-\infty,-1]\cup[1,\infty)$ and there are no imbedded eigenvalues or resonances in the essential spectrum, the discrete spectrum is of the form $\{0, i\mu, -i\mu\}$ where $\mu>0$ with $\pm i\mu$ both simple eigenvalues, the root-space at 0 is of dimension two, and the thresholds ±1 are neither eigenvalues nor resonances. In explicit form, the root space is spanned by*

$$\xi_0 = \begin{pmatrix} Q \\ -Q \end{pmatrix}, \qquad \eta_0 = \begin{pmatrix} \partial_\alpha Q \\ \partial_\alpha Q \end{pmatrix} \tag{B.2}$$

and one has $\mathcal{H}\xi_0 = 0, \mathcal{H}^2\eta_0 = 0$. Let $\mathcal{H}G_\pm = \mp i\mu G_\pm$ with the normalization $\|G_\pm\|_2 = 1$. Then the eigenfunctions $G_\pm$ are exponentially decaying and of the form $G_\pm = \binom{g_\pm}{\bar{g}_\pm}$.

Proof. The description of the root space of $\mathcal{H}$ goes back to [60]. The imaginary spectrum was identified in [51], and for the exponential decay of the corresponding eigenfunctions see [32]. See Grillakis [26], [27] for more on the discrete spectrum. All these results are based on purely analytical arguments. The fact that $\mathcal{H}$ does not have embedded eigenvalues in the essential spectrum was shown in [41], assisted by some numerical computations. Their proof also implies that there are no non–zero eigenvalues in the gap $[-1,1]$, and that the thresholds are not resonances. Alternatively, the latter two facts also follow by the analytical arguments in [51] combined with the numerics in [16]. $\square$

Next, we present a result for non-selfadjoint Schrödinger evolutions which originates in [4] (in fact, Beceanu proves a stronger result in Lorentz spaces). Let $S = \bigcap_{p,q} L^p_t(\mathbb{R}^+, L^q_x)$ be the Strichartz space with $2\le p\le\infty$, $2\le q\le 6$, and $\frac{2}{p}+\frac{3}{q}=\frac{3}{2}$, and let S^* be its dual.

Lemma B.2. *Let $a\in L^\infty(\mathbb{R})$ satisfy $\|a\|_\infty < c_0$ for some small absolute constant c_0. The solution $\Psi\in C(\mathbb{R};L^2(\mathbb{R}^3))\cap C^1(\mathbb{R};H^{-2}(\mathbb{R}^3))$ of the problem*

$$i\partial_t\Psi + \mathcal{H}\Psi + a(t)\sigma_3 P_c\,\Psi = F \in S^*, \qquad \Psi(0)=\Psi_0\in L^2(\mathbb{R}^3) \tag{B.3}$$

where P_c is the projection corresponding to the essential spectrum of $\mathcal{H}$, obeys the Strichartz estimates

$$\|P_c\Psi\|_S \lesssim \|\Psi_0\|_{L^2(\mathbb{R}^3)} + \|F\|_{S^*} \tag{B.4}$$

Furthermore, if $\Psi_0\in H^1$, then

$$\|\nabla P_c\Psi\|_S \lesssim \|\Psi_0\|_{H^1(\mathbb{R}^3)} + \|\nabla F\|_{S^*} \tag{B.5}$$

[5]The only change to Proposition B.1 is that $\mathcal{H}$ has a root-space of dimension eight rather than two.

Finally, one has scattering: there exists $\Psi_\infty \in H^1$ such that

$$P_c\Psi(t) = e^{i\sigma_3 \int_0^t (-\Delta+1+a(s))\, ds}\Psi_\infty + o(1) \tag{B.6}$$

in H^1 as $t \to \infty$.

Proof. We follow [4]. Clearly, the proof should be perturbative in a by nature, with $a = 0$ being the nontrivial statement that Strichartz estimates (including the endpoint) hold for the equation

$$i\partial_t\Psi + \mathcal{H}\Psi = F$$

However, the latter has been established by several authors, see for example [4, Theorem 1.3] and [15]. Due to the lack of any physical localization of the $a(t)$ term, the perturbative analysis is nontrivial. On the other hand, note that any perturbation of the form $a(t)\chi(x)$ where the multiplier χ is bounded $L^6(\mathbb{R}^3) \to L^{\frac{6}{5}}(\mathbb{R}^3)$ can be taken to the right-hand side by virtue of the endpoint Strichartz estimate.

To commence with the actual argument, consider the following auxiliary equation, with arbitrary but fixed $\delta > 0$, and $P_d = \mathrm{Id} - P_c$:

$$i\partial_t Z + \mathcal{H}P_c Z + i\delta P_d Z + a(t)\sigma_3\, P_c Z = F \tag{B.7}$$

with data $Z(0) = \Psi_0$. We claim the Strichartz estimates for general data $Z(0)$,

$$\|Z\|_S \lesssim \|Z(0)\|_{L^2(\mathbb{R}^3)} + \|F\|_{S^*} \tag{B.8}$$

If so, then $\tilde{Z} := P_c Z$ satisfies $\tilde{Z}(0) = P_c\Psi_0$ and

$$i\partial_t\tilde{Z} + \mathcal{H}\tilde{Z} + a(t)\sigma_3\,\tilde{Z} = P_c F + a(t)[\sigma_3, P_c]\tilde{Z} \tag{B.9}$$

which is the same as the P_c projection of (B.3). Thus, $P_c\Psi = \tilde{Z}$ and (B.8) implies (B.4). Let $A(t) = \int_0^t a(s)\, ds$ and write $U(t) = e^{iA(t)\sigma_3}$, $Z(t) = U(t)\Phi$. Then (B.7) becomes

$$i\partial_t\Phi + U^{-1}(\mathcal{H}P_c + i\delta P_d)U\Phi = U^{-1}F + a(t)U^{-1}\sigma_3\, P_d U\Phi =: F_1 \tag{B.10}$$

or, with $\Phi(0) = Z(0)$,

$$i\partial_t\Phi + \mathcal{H}_0\Phi = -U^{-1}(V - \mathcal{H}P_d + i\delta P_d)U\Phi + F_1 \tag{B.11}$$

The matrix operators $\mathcal{H}_0, V$ are defined via:

$$\mathcal{H}_0 = \begin{bmatrix} -\Delta+1 & 0 \\ 0 & \Delta-1 \end{bmatrix}, \qquad \mathcal{H} = \mathcal{H}_0 + V$$

Choose a smooth, exponentially decaying matrix potential V_2 which is invertible and such that the operator

$$V_1 := (V - \mathcal{H}P_d + i\delta P_d)V_2^{-1}$$

is bounded from $L^p \to L^q$ for any $1 \le p, q \le \infty$. In other words,

$$V_1V_2 = V - \mathcal{H}P_d + i\delta P_d \tag{B.12}$$

with V_1, V_2 being bounded from $L^p \to L^q$ for any $1 \le p, q \le \infty$. By Duhamel the solution to (B.11) is

$$\Phi(t) = e^{it\mathcal{H}_0}\Phi(0) - i\int_0^t e^{i(t-s)\mathcal{H}_0}[-U^{-1}V_1V_2U\Phi + F_1](s)\,ds \tag{B.13}$$

Applying $U(t)$ to both sides yields, since U commutes with the propagator of $\mathcal{H}_0$,

$$Z(t) = U(t)e^{it\mathcal{H}_0}Z(0) + i\int_0^t e^{i(t-s)\mathcal{H}_0}[U(t)U^{-1}(s)V_1V_2Z(s) - U(t)F_1(s)]\,ds \tag{B.14}$$

We introduce the operators

$$\begin{aligned} T_0F(t) &:= V_2\int_0^t e^{i(t-s)\mathcal{H}_0}V_1F(s)\,ds \\ \tilde{T}_0F(t) &:= V_2\int_0^t e^{i(t-s)\mathcal{H}_0}U(t)U(s)^{-1}V_1F(s)\,ds \end{aligned} \tag{B.15}$$

By the Strichartz estimates for the free equation, $T_0, \tilde{T}_0$ are bounded on $L^2_{t,x}$. By (B.14),

$$V_2Z = i\tilde{T}_0V_2Z + V_2U(t)e^{it\mathcal{H}_0}Z(0) - iV_2U(t)\int_0^t e^{i(t-s)\mathcal{H}_0}F_1(s)\,ds \tag{B.16}$$

Suppose

$$(\mathrm{Id} - i\tilde{T}_0)^{-1} \;:\; L^2_{t,x} \to L^2_{t,x} \tag{B.17}$$

as a bounded operator. Then (B.16) implies via the endpoint Strichartz estimate, see (B.10),

$$\|V_2Z\|_{L^2_{t,x}} \lesssim \|Z(0)\|_2 + \|F_1\|_{S^*} \lesssim \|Z(0)\|_2 + \|F\|_{S^*} + c_0\|V_2Z\|_{L^2_{t,x}} \tag{B.18}$$

To pass to the final estimate we wrote $P_dZ = P_dV_2^{-1}V_2Z$ and used that $P_dV_2^{-1}$ is bounded by construction. Inserting the resulting bound on $\|V_2Z\|_{L^2_{t,x}}$ back into (B.14) yields the desired estimate (B.8). It therefore remains to prove (B.17) which will follow from

$$(\mathrm{Id} - iT_0)^{-1} \;:\; L^2_{t,x} \to L^2_{t,x} \tag{B.19}$$

provided we can show that $\|T_0 - \tilde{T}_0\| \ll 1$ in the operator norm on $L^2_{t,x}$. This, however, follows from the pointwise dispersive estimate on $e^{it\mathcal{H}_0}$ which yields

$$\|V_2e^{i(t-s)\mathcal{H}_0}(U(t)U(s)^{-1} - 1)V_1F(s)\|_2 \lesssim \|a\|_\infty^{\frac14}\langle t-s\rangle^{-\frac54}\|F(s)\|_2 \tag{B.20}$$

Thus, we have reduced ourselves to proving (B.19). We introduce

$$T_1F(t) := V_2\int_0^t e^{i(t-s)\mathcal{H}P_c - (t-s)\delta P_d}V_1F(s)\,ds \tag{B.21}$$

As for the meaning of T_1, first note that due to commutativity,

$$\begin{aligned} e^{it\mathcal{H}P_c - t\delta P_d} &= e^{it\mathcal{H}P_c}e^{-t\delta P_d} \\ &= \big(e^{it\mathcal{H}}P_c + P_d\big)e^{-t\delta P_d} = e^{it\mathcal{H}}P_c + e^{-t\delta P_d}P_d \end{aligned} \tag{B.22}$$

satisfies Strichartz estimates as in (B.4), see [3], [4], as well as [15]. Second, the solution to

$$i\partial_t Z + \mathcal{H} P_c Z + i\delta P_d Z = 0 \tag{B.23}$$

can be written in two ways:

$$\begin{aligned} Z(t) &= e^{it\mathcal{H}P_c - t\delta P_d} Z(0) \\ Z(t) &= e^{it\mathcal{H}_0} Z(0) + i\int_0^t e^{i(t-s)\mathcal{H}_0}(V - \mathcal{H}P_d + i\delta P_d) Z(s)\, ds \end{aligned} \tag{B.24}$$

Thus, one further has

$$e^{it\mathcal{H}P_c - t\delta P_d} Z(0) = e^{it\mathcal{H}_0} Z(0) + i\int_0^t e^{i(t-s)\mathcal{H}_0}(V - \mathcal{H}P_d + i\delta P_d) e^{is\mathcal{H}P_c - s\delta P_d} Z(0)\, ds \tag{B.25}$$

Therefore, we conclude that

$$\begin{aligned} T_0 T_1 F(t) &= V_2 \int_0^t \int_0^s e^{i(t-s)\mathcal{H}_0}(V - \mathcal{H}P_d + i\delta P_d) e^{i(s-s_1)\mathcal{H}P_c - (s-s_1)\delta P_d} V_1 F(s_1)\, ds_1 ds \\ &= -iV_2 \int_0^t (e^{i(t-s_1)\mathcal{H}P_c - (t-s_1)\delta P_d} - e^{i(t-s_1)\mathcal{H}_0}) V_1 F(s_1)\, ds_1 \end{aligned} \tag{B.26}$$

or $T_0T_1 + i(T_1 - T_0) = 0$ which implies that

$$(\mathrm{Id} - iT_0)(\mathrm{Id} + iT_1) = \mathrm{Id} \tag{B.27}$$

On the other hand,

$$\begin{aligned} T_1 T_0 F(t) &= V_2 \int_0^t \int_0^s e^{i(t-s)\mathcal{H}P_c - (t-s)\delta P_d}(V - \mathcal{H}P_d + i\delta P_d) e^{i(s-s_1)\mathcal{H}_0} V_1 F(s_1)\, ds_1 ds \\ &= iV_2 \int_0^t \int_{s_1}^t \partial_s \big[e^{i(t-s)\mathcal{H}P_c - (t-s)\delta P_d}\ e^{i(s-s_1)\mathcal{H}_0} \big]\, ds\, V_1 F(s_1)\, ds_1 \\ &= -iV_2 \int_0^t (e^{i(t-s_1)\mathcal{H}P_c - (t-s_1)\delta P_d} - e^{i(t-s_1)\mathcal{H}_0}) V_1 F(s_1)\, ds_1 \end{aligned} \tag{B.28}$$

whence $T_1T_0 + i(T_1 - T_0) = 0$ which implies that

$$(\mathrm{Id} + iT_1)(\mathrm{Id} - iT_0) = \mathrm{Id} \tag{B.29}$$

These identities hold in the algebra of bounded operators on $L^2_{t,x}$, as justified by the endpoint Strichartz estimates. Thus (B.19) holds and (B.4) follows. For (B.5) one applies a gradient to (B.3).

From (B.13), we obtain the scattering of Φ in the following sense:

$$\Phi(t) = e^{it\mathcal{H}_0}\Phi_\infty + o(1) \quad t \to \infty \tag{B.30}$$

in H^1 for some $\Phi_\infty \in H^1$. Thus,

$$P_c\Psi(t) = P_c U(t)\big[e^{it\mathcal{H}_0}\Phi_\infty + o(1)\big] = e^{iA(t)\sigma_3} e^{it\mathcal{H}_0}\Phi_\infty + o(1) \quad t \to \infty \tag{B.31}$$

in H^1, as claimed. $\square$

Appendix C. Some radial Sobolev inequalities

For the reader's convenience, we prove some elementary Sobolev-type inequalities for radial functions used in this paper. For any radial smooth function $u(x) = u(r)$ with compact support on $\mathbb{R}^3$, we have for any $R > 0$,

$$\sup_{r>R} |u(r)|^2 \leq \int_R^\infty |2uu_r| dr \leq 2\|u_r\|_{L_r^2(R,\infty)}\|u\|_{L_r^2(R,\infty)}, \tag{C.1}$$

by Cauchy-Schwarz, where L_r^2 denotes the L^2 space for $r \in \mathbb{R}$ without any weight. Also by partial integration,

$$\int_R^\infty |u(r)|^2 dr \leq \int_R^\infty 2|uu_r(r-R)| dr \leq 2\|u_r\|_{L^2(R,\infty)}\|ru\|_{L_r^2(R,\infty)}. \tag{C.2}$$

Plugging the latter estimate into the former one obtains

$$\|u\|_{L_r^\infty(R,\infty)} \leq 2\|u_r\|_{L_r^2(R,\infty)}^{3/4}\|ru\|_{L_r^2(R,\infty)}^{1/4}. \tag{C.3}$$

Combining the above two estimates yields

$$\int_R^\infty |u(r)|^4 dr \leq \|u\|_{L_r^\infty(R,\infty)}^2\|u\|_{L_r^2(R,\infty)}^2 \leq 8\|u_r\|_{L_r^2(R,\infty)}^{5/2}\|ru\|_{L_r^2(R,\infty)}^{3/2}, \tag{C.4}$$

and

$$\int_R^\infty |u(r)|^4 r^2 dr \leq \|u\|_{L_r^\infty(R,\infty)}^2\|ru\|_{L_r^2(R,\infty)}^2 \leq 4\|u_r\|_{L_r^2(R,\infty)}^{3/2}\|ru\|_{L_r^2(R,\infty)}^{5/2}. \tag{C.5}$$

Interpolating the above two, we obtain

$$\int_R^\infty |u(r)|^4 r dr \leq 4\sqrt{2}\int_R^\infty |u_r(r)|^2 dr \int_R^\infty |u(r)|^2 r^2 dr. \tag{C.6}$$

By a density argument this estimate extends to any radial $u \in H^1(\mathbb{R}^3)$. Furthermore,

$$\sup_{r>R} |u(r)|^2 \leq \int_R^\infty |2uu_r| dr \leq 2R^{-2}\|ru_r\|_{L_r^2(R,\infty)}\|ru\|_{L_r^2(R,\infty)}, \tag{C.7}$$

and so

$$\int_R^\infty |u(r)|^4 r^2 dr \leq \|u\|_{L_r^\infty(R,\infty)}^2\|ru\|_{L_r^2(R,\infty)}^2 \leq 2R^{-2}\|ru_r\|_{L_r^2(R,\infty)}\|ru\|_{L_r^2(R,\infty)}^3. \tag{C.8}$$

Appendix D. Table of Notation

symbols	description	defined in
$M(u), E(u)$	mass and energy	(1.2)
$\mathcal{H}$, $\mathcal{H}^\varepsilon$, $\mathcal{H}_\alpha^\varepsilon$, $\mathcal{H}_{(\delta)}$	energy space and subsets	(1.3), (1.4), (3.41)
Q, Q_α, $\mathcal{S}$, $\mathcal{S}_\alpha$	solitons, soliton manifolds	Section 1
Q'_α	derivative of Q_α in α	(2.13)
$J(u)$, $K(u)$, $G(u)$, $I(u)$	action and derived functionals	(2.3), (2.5)
$\mathcal{L}$, L_+, L_-	$\mathbb{R}$-linear linearized Hamiltonian	(2.9)
$\mathfrak{G}_\pm$, $\lambda_\pm$	unstable/stable modes of $i\mathcal{L}$	(2.12)
$C(w)$	super-quadratic part of $J(u) - J(Q)$	(2.21)
α, θ	modulation parameters	(3.1)
$\|\cdot\|_E^2$	linearized energy norm	(3.14)

$d_Q(u)$	nonlinear distance to $\mathcal{S}_1$	(3.15)
δ_E, δ_X	smallness scales for ejection	Lemma 3.3
$\mathfrak{S}(u)$	continuation of $\operatorname{sign}(K(u))$ to $\mathcal{H}_{(\delta)}$	Lemma 3.5
δ_S	smallness scale needed for $\mathfrak{S}(u)$	Lemma 3.5
ε_*, R_*	smallness scales for 1-pass theorem	Theorem 4.1, (4.3)
v_n^j, u_n^j, γ_n^k	Bahouri-Gerard decomposition	Section 5
$\mathcal{H}(\alpha,\gamma)$	matrix Hamiltonian	(7.1)
P_c, P_0, $P_\pm$	Riesz projections for $\mathcal{H}$	(7.22)
$\mathcal{M}$, $\mathcal{M}_{\alpha,\gamma}$, $\mathcal{M}_{\mathcal{S}}$	center stable manifolds	Section 7
ξ_0, η_0	root modes of matrix Hamiltonian	(B.2)
$G_\pm$	discrete imaginary modes of $\mathcal{H}$	Proposition B.1

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Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `n-kenji@math.kyoto-u.ac.jp`

Department of Mathematics, The University of Chicago, Chicago, IL 60615, U.S.A.
E-mail address: `schlag@math.uchicago.edu`